BILL AND EMILY RECORD LECTURE!!!!
Problems with a Point: Exploring Math and Computer Science
Authors:
William Gasarch
Clyde Kruskal
How This Book Came to Be
In 2003 Lance Fortnow started Complexity Blog. In 2007 Bill Gasarch joined and it was a co-blog. In 2015 various book publishers asked us Can you make a book out of your blog? Lance declined but Bill said YES.
Book’s Origin

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- do some math to underscore those points
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and made those into chapters.
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**Caveat:** Not every chapter is quite like that.
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**Caveat:** Not every chapter is quite like that.
To quote Ralph Waldo Emerson

  *A foolish consistency is the hobgoblin of small minds.*
Possible Subtitles

Problems with a **Point** needed a subtitle.

I proposed **Problems with a Point: Mathematical Musing and Math to make those Musings Magnificent**

The publisher said **NO!**

I proposed **Problems with a Point: Mathematical Meditations and Computer Science Cogitations**

The publisher wisely decided to be less cute and more informative: **Problems with a Point: Exploring Math and Computer Science**
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Problems with a Point: Exploring Math and Computer Science
Clyde Joins the Project!

After some samples of Bill’s writing the publisher said
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**Please Procure People to Polish Prose and Proofs of Problems with a Point**

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Clyde Kruskal became a co-author.
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Now onto some samples of the book!
Point: Students Can Give Strange Answers
The Paint Can Problem

From the Year 2000 Maryland Math Competition:
There are 2000 cans of paint. Show that at least one of the following two statements is true:

▶ There are at least 45 cans of the same color.
▶ There are at least 45 cans that are different colors.

Work on it in groups! Prove a General Theorem.

Answer:
If there are 45 different colors of paint then we are done. Assume there are \( \leq 44 \) different colors. If all colors appear \( \leq 44 \) times then there are \( 44 \times 44 = 1936 \) < 2000 cans of paint, a contradiction.

Note: this was Problem 1, which is supposed to be easy and indeed 95% got it right. What about the other 5%? Next slide.
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**ANSWER:**

Paint cans are grey. Hence there are all the same color. Therefore there are 2000 cans that are the same color.
One of the Wrong Answers. Or is it?

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What do you think:

- Thats just stupid. 0 points.
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▶ Thats just stupid. 0 points.
▶ Question says *cans of the same color.* ... The full 30 pts.
▶ Not only does he get 30 points, but everyone else should get 0.
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If you look at a paint color really really carefully there will be differences. Hence, even if two cans seem to both be (say) RED, they are really different. Therefore there are 2000 cans of different colors.
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A Triangle Problem

From the year 2007 Maryland Math Competition.

QUESTION Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.
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**Note** I think I was assigned to grade it since it looks like the kind of problem I would make up, even though I didn’t. It was problem 5 (out of 5) and was hard. About 100 students tried it, 8 got full credit, 10 got partial credit.
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Funny Answer One
All the vertices are red because I can make them whatever color I want. I can also write at a 30 degree angle to the bottom of this paper (The students answer was written at a 30 degree angle to the bottom of the paper.) if thats what I feel like doing at the moment. Just like $2 + 2 = 5$ if thats what my math teacher says. Math is pretty subjective anyway.
Was Student One Serious?

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**Theorem** The students is not serious.
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**Theorem** The student is not serious.

**Proof** Assume, by contradiction, that they are serious. Then they really think math is subjective. Hence they don't really understand math. Hence they would not have done well enough on Part I to qualify for Part II. But they took Part II. Contradiction.
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I like to think that we live in a world where points are not judged by their color, but by the content of their character. Color should be irrelevant in the plane. To prove that there exists a group of points where only one color is acceptable is a reprehensible act of bigotry and discrimination.
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Was Student Two Serious?
QUESTION Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

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Was Student Two Serious? Yes.
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Was Student Two Serious? Yes. About Justice!
Each point in the plane is colored either red or green. Let ABC be a fixed triangle. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

Fix a 2-coloring of the plane.
There are 3 equally-spaced mono points on x-axis

Proof Clearly there are two points on the x-axis of the same color: $p_1, p_2$ are RED. If $p_3$, the midpoint of $p_1, p_2$, is RED then $p_1, p_3, p_2$ are all RED. DONE. Hence we assume $p_3$ is GREEN.
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**Proof** Clearly there are two points on the $x$-axis of the same color: $p_1, p_2$ are RED. If $p_3$, the midpoint of $p_1, p_2$, is RED then $p_1, p_3, p_2$ are all RED. DONE. Hence we assume $p_3$ is GREEN.

Let $p_4$ be such that $|p_1 - p_4| = |p_2 - p_1|$. If $p_4$ is RED then $p_4, p_1, p_2$ are all RED. DONE. Hence we assume $p_4$ is GREEN.
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Let $p_5$ be such that $|p_5 - p_2| = |p_2 - p_1|$. If $p_5$ is RED then $p_1, p_2, p_5$ are all RED. DONE. Hence we assume $p_5$ is GREEN.
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Only case left $p_3, p_4, p_5$ are all GREEN. DONE.
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Only case left $p_3, p_4, p_5$ are all GREEN. DONE.
$P, Q, R$ are RED.

If $T$ or $U$ or $S$ are RED then get RED Triangle similar to ABC.

If not then ALL of $T, U, S$ are GREEN, so get GREEN triangle similar to ABC.
Point: What is a Pattern?
Simple Functions

Bill assigned the following in Discrete Math: For each of the following sequences find a simple function $A(n)$ such that the sequence is $A(1), A(2), A(3), \ldots$

1. 10, -17, 24, -31, 38, -45, 52, \ldots
2. -1, 1, 5, 13, 29, 61, 125, \ldots
3. 6, 9, 14, 21, 30, 41, 54, \ldots

Caveat: These are NOT trick questions. Work on it in groups.
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Caveat: These are NOT trick questions.

Work on it in groups.

1. 10, -17, 24, -31, 38, -45, 52, … $A(n) = (-1)^{n+1}(7n + 3)$.
2. -1, 1, 5, 13, 29, 61, 125, … $A(n) = 2^n - 3$.
3. 6, 9, 14, 21, 30, 41, 54, … $A(n) = n^2 + 5$. 
A Student asks — What is a Simple Function?

One student, in earnest, emailed Bill the following:

*We never defined **Simple Function** in class so I went to Wikipedia.*
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I told him NO— all I wanted is an easy-to-describe function.
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I should have told him to use that def to see what he did.
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I should have told him to use that def to see what he did.
The student got the first one right, but left the other two blank.
When Do Patterns Hold?

The last question brings up the question of when patterns do and don’t hold. We looked for cases where a pattern did not hold.
First Non-Pattern: $n$ Points on a circle

What is the max number of regions formed by connecting every pair of $n$ points on a circle. For $n = 1, 2, 3, 4, 5$:

Based on this data what guess is tempting? $2n - 1$.

But for $n = 6$, the number of regions is only 31.

The actual number of regions for $n$ points is $(\binom{n}{4}) + (\binom{n}{2}) + 1$. 


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- For $n = 1$: 1 region
- For $n = 2$: 2 regions
- For $n = 3$: 4 regions
- For $n = 4$: 8 regions
- For $n = 5$: 16 regions

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![Diagram of points on a circle connected by lines]

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The actual number of regions for $n$ points is $\binom{n}{4} + \binom{n}{2} + 1$. 
Second Non-Pattern: Borwein Integrals

\[
\int_{0}^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}
\]

\[
\int_{0}^{\infty} \frac{\sin x \sin \frac{x}{3}}{x} = \frac{\pi}{2}
\]

\[
\vdots
\]

\[
\int_{0}^{\infty} \frac{\sin x \sin \frac{x}{3} \sin \frac{x}{5} \sin \frac{x}{7} \sin \frac{x}{9} \sin \frac{x}{11} \sin \frac{x}{13}}{x} = \frac{\pi}{2}
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\[
\begin{align*}
\int_0^\infty \frac{\sin x}{x} = & \quad \frac{\pi}{2} \\
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: & \\
\int_0^\infty \frac{\sin x \sin \frac{x}{3} \sin \frac{x}{5} \sin \frac{x}{7} \sin \frac{x}{9} \sin \frac{x}{11} \sin \frac{x}{13} \sin \frac{x}{15}}{x} = & \quad \frac{\pi}{2}
\end{align*}
\]

But

\[
\int_0^\infty \frac{\sin x \sin \frac{x}{3} \sin \frac{x}{5} \sin \frac{x}{7} \sin \frac{x}{9} \sin \frac{x}{11} \sin \frac{x}{13} \sin \frac{x}{15}}{x} = \frac{\pi}{2}
\]

\[
\frac{4678079247134407386537864469\pi}{935615849440640907310521750000} \approx 0.9999999999852937186 \times \frac{\pi}{2}
\]
Why the breakdown at 15?

Because

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1$$

but

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15} > 1.$$
Another Non-Pattern: More Borwein Integrals

\[ \int_0^\infty 2 \cos(x) \frac{\sin x}{x} = \frac{\pi}{2} \]
Another Non-Pattern: More Borwein Integrals

\[
\int_0^\infty 2 \cos(x) \frac{\sin x}{x} = \frac{\pi}{2}
\]

\[
\int_0^\infty 2 \cos(x) \frac{\sin x \sin \frac{x}{3}}{x \frac{x}{3}} = \frac{\pi}{2}
\]
Another Non-Pattern: More Borwein Integrals

\[
\int_{0}^{\infty} 2 \cos(x) \frac{\sin x}{x} = \frac{\pi}{2}
\]

\[
\int_{0}^{\infty} 2 \cos(x) \frac{\sin x \sin \frac{x}{3}}{x} = \frac{\pi}{2}
\]

\vdots
Another Non-Pattern: More Borwein Integrals

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\int_0^\infty 2 \cos(x) \frac{\sin x}{x} = \frac{\pi}{2}
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\]

\[
\vdots
\]

\[
\int_0^\infty 2 \cos(x) \frac{\sin x \sin \frac{x}{3} \ldots \sin \frac{x}{111}}{x} = \frac{\pi}{2}
\]
Another Non-Pattern: More Borwein Integrals

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\[
\int_0^\infty 2 \cos(x) \frac{\sin x \sin \frac{x}{3}}{x \frac{x}{3}} = \frac{\pi}{2}
\]

\[
\int_0^\infty 2 \cos(x) \frac{\sin x \sin \frac{x}{3} \cdots \sin \frac{x}{111}}{x \frac{x}{3} \cdots \frac{x}{111}} = \frac{\pi}{2}
\]

\[
\int_0^\infty 2 \cos(x) \frac{\sin x \sin \frac{x}{3} \cdots \sin \frac{x}{113}}{x \frac{x}{3} \cdots \frac{x}{113}} < \frac{\pi}{2}
\]
Why the breakdown at 113?

Because

\[ \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{111} < 2 \]

but

\[ \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{113} > 2. \]
Computers to FIND proofs vs Computers to DO Proofs
Colorings and Square Differences

The following are all true:

1. There exists a number $W_2$ such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.
Colorings and Square Differences

The following are all true:

1. There exists a number $W_2$ such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.
2. There exists a number $W_3$ such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.
Colorings and Square Differences

The following are all true:

1. There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

2. There exists a number $W_3$ such that, for all 3-colorings of \{1, \ldots, W_3\} there exists 2 nums, square-apart, same color.

3. There exists a number $W_4$ such that, for all 4-colorings of \{1, \ldots, W_4\} there exists two nums, square-apart, same color.

The proofs in the literature of these theorems give ENORMOUS bounds on $W_2, W_3, W_4, W_c$. We look at easier proofs with two points in mind:

- Would they be good questions on a HS math competition?
- What is the role of Computers in these proofs?
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1. There exists a number $W_2$ such that, for all 2-colorings of
   $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.
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   $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.
3. There exists a number $W_4$ such that, for all 4-colorings of
   $\{1, \ldots, W_4\}$ there exists two nums, square-apart, same color.
4. For all $c$ there exists a number $W_c$ . . . .
Colorings and Square Differences

The following are all true:

1. There exists a number $W_2$ such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.
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4. For all $c$ there exists a number $W_c$ . . .

The proofs in the literature of these theorems give EEEEEEEEEENORMOUS bounds on $W_2, W_3, W_4, W_c$. We look at easier proofs with two points in mind:

- Would they be good questions on a HS math competition?
- What is the role of Computers in these proofs?
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \( \{1, \ldots, W_2\} \) there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize $W_2$. 
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize $W_2$.

Let $COL$ be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings $R$ and $B$. We can assume $COL(1) = R$. 
There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize $W_2$.

Let $\text{COL}$ be a 2-coloring of \{1, 2, 3, \ldots\} with colorings $R$ and $B$. We can assume $\text{COL}(1) = R$. Since 1 is a square $\text{COL}(2) = B$. 

AH-HA: $\text{COL}(1) = \text{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.

AH-HA: $RBRB$ shows that $W_2 \leq 5$. So $W_2 = 4$. Upshot Could be easy HS Math Comp Prob. No computer used.
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize $W_2$.

Let COL be a 2-coloring of \{1, 2, 3, \ldots\} with colorings $R$ and $B$. We can assume COL(1) = $R$.
Since 1 is a square COL(2) = $B$.
Since 1 is a square COL(3) = $R$. 

AH-HA: COL(1) = COL(5) and 5 − 1 = 4 = 2². So $W_2 \leq 5$.

AH-HA: RBRB shows that $W_2 \leq 5$.

So $W_2 = 4$.

Upshot: Could be easy HS Math Comp Prob. No computer used.
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \{1,\ldots, W_2\} there exists 2 nums, square-apart, same color.

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Since 1 is a square $\text{COL}(2) = B$.
Since 1 is a square $\text{COL}(3) = R$.
Since 1 is a square $\text{COL}(4) = B$. 

AH-HA: $\text{COL}(1) = \text{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.

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Since 1 is a square $COL(3) = R$.

Since 1 is a square $COL(4) = B$.

Since 1 is a square $COL(5) = R$.

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Since 1 is a square COL(3) = $R$.

Since 1 is a square COL(4) = $B$.

Since 1 is a square COL(5) = $R$.

AH-HA: COL(1) = COL(5) and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$. 
There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

**Work on in groups and try to minimize $W_2$.**

Let COL be a 2-coloring of \{1, 2, 3, \ldots\} with colorings $R$ and $B$. We can assume COL(1) = $R$.

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Since 1 is a square COL(3) = $R$.

Since 1 is a square COL(4) = $B$.

Since 1 is a square COL(5) = $R$.

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**Upshot**

Could be easy HS Math Comp Prob. No computer used.
2-colorings and Square Differences

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Upshot  Could be easy HS Math Comp Prob. No computer used.
3-colorings and Square Differences

There exists a number $W_3$ such that, for all 3-colorings of \{1, \ldots, W_3\} there exists 2 nums, square-apart, same color.
3-colorings and Square Differences

There exists a number $W_3$ such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize $W_3$. 

Figure: $\text{COL}(x) = \text{COL}(x + 41)$
3-colorings and Square Differences

There exists a number $W_3$ such that, for all 3-colorings of \{1, \ldots, W_3\} there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize $W_3$.

Figure: $\text{COL}(x) = \text{COL}(x + 41)$
Since $\text{COL}(x) = \text{COL}(x + 41) \ldots$

Use $\text{COL}(x) = \text{COL}(x + 41)$ to finish the proof and find upper bound on $W_3$. 

So 1 and 41 are a square apart and the same color.

$W_3 \leq 1 + 41^2 = 1682$

Can we get better bound on $W_3$?
Since $\text{COL}(x) = \text{COL}(x + 41) \ldots$

Use $\text{COL}(x) = \text{COL}(x + 41)$ to finish the proof and find upper bound on $W_3$.

$$\text{COL}(1) = \text{COL}(1+41) = \text{COL}(1+2\times41) = \cdots = \text{COL}(1+41\times41)$$
Since $\text{COL}(x) = \text{COL}(x + 41)$ . . .

Use $\text{COL}(x) = \text{COL}(x + 41)$ to finish the proof and find upper bound on $W_3$.

$\text{COL}(1) = \text{COL}(1+41) = \text{COL}(1+2\times41) = \cdots = \text{COL}(1+41\times41)$

So 1 and $41^2$ are a square apart and the same color.

$W_3 \leq 1 + 41^2 = 1682$
Since $\text{COL}(x) = \text{COL}(x + 41) \ldots$

Use $\text{COL}(x) = \text{COL}(x + 41)$ to finish the proof and find upper bound on $W_3$.

$$\text{COL}(1) = \text{COL}(1 + 41) = \text{COL}(1 + 2 \times 41) = \cdots = \text{COL}(1 + 41 \times 41)$$

So 1 and $41^2$ are a square apart and the same color.

$W_3 \leq 1 + 41^2 = 1682$

Can we get better bound on $W_3$?
Better Bound on $W_3$

Figure: If $x \geq 10$ then $\text{COL}(x) = \text{COL}(x + 7)$, so $W_3 \leq 59$
Reflection on $W_3$, $W_4$

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006: Show that for all 3-colorings of \{1,...,2006\} there exists 2 numbers that are a square apart that are the same color.

2. 240 took exam, 40 tried this problem, 10 got it right.


4. Is there a HS-proof that $W_4$ exists? Bill wanted to put this problem on the next HS exam to find out. He was (wisely) told NO.

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4. **Is there a HS-proof that $W_4$ exists?** Bill wanted to put this problem on the next HS exam to find out. He was (wisely) told **NO**.

5. The question still remains: Is there a HS proof that $W_4$ exists?
Reflection on $W_3, W_4$

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006: Show that for all 3-colorings of \{1, \ldots, 2006\} there exists 2 numbers that are a square apart that are the same color.

2. 240 took exam, 40 tried this problem, 10 got it right.


4. **Is there a HS-proof that $W_4$ exists?** Bill wanted to put this problem on the next HS exam to find out. He was (wisely) told **NO**.

$W_4$ Exists: $\text{COL}(x) = \text{COL}(x + 290, 085, 290)$
Reflection on $W_4$

1. Zach's proof shows $W_4 \leq 1 + 299,085,290^2$.
   
   **PRO**: Proof is easy to verify
   
   **CON**: Number is large, proof does not generalize to $W_5$.

2. The classical proof.
   
   **PRO**: Gives bounds for $W_c$.
   
   **CON**: Bounds are GINORMOUS, even for $W_2$.

3. A Computer Search showed that $W_4 = 58$.
   
   **PRO**: Get exact value.
   
   **CON**: not human-verifiable. Does not generalize to $W_5$.

Which do you prefer?
Reflection on $W_4$

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Which do you prefer?
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