Loaded Dice
Fair Dice Yield Unfair Sums

**Fair Die:**

$$\Pr(1)=\Pr(2)=\Pr(3)=\Pr(4)=\Pr(5)=\Pr(6) = \frac{1}{6} \sim 0.167$$

Roll TWO of them.

$$\Pr(\text{Sum}=2)=\frac{1}{36} \text{ (This is Min Pr(Sum))}$$
$$\Pr(\text{Sum}=7)=\frac{1}{6}. \text{ (This is Max Pr(Sum))}$$
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**Sums are Unfair!**
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Sums are Unfair!

How Unfair?: \[ \frac{1}{6} - \frac{1}{36} \approx 0.139 \text{ unfair.} \]
What are Loaded Dice?

**Def** A *Die* is a 6-tuple \((p_1, p_2, p_3, p_4, p_5, p_6)\) such that \(0 \leq p_i \leq 1\) and \(\sum_{i=1}^{6} p_i = 1\).
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**Our Questions:**
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**VOTE**
1) There exists a way to load dice so that all sums are prob \(\frac{1}{11}\).
2) There is no way to load dice so that all the sums are prob \(\frac{1}{11}\).
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What Do You Think?

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1) There exists a way to load dice so that all sums are prob \(\frac{1}{11}\).
2) There is no way to load dice so that all the sums are prob \(\frac{1}{11}\).

No such dice can exist!
Assume that are dice that yield fair sums. Let $(p_1, \ldots, p_6)$ and $(q_1, \ldots, q_6)$ be those dice.

**KEY:**

\[(p_1x + p_2x^2 + \cdots + p_6x^6)(q_1x + q_2x^2 + \cdots + q_6x^6)\]

Coefficient of $x^5$ is

\[p_1 q_4 + p_2 q_3 + p_3 q_2 + p_4 q_1 = \text{Prob}(\text{sum} = 5)\]

Coefficient of $x^i$ is $\text{Prob}(\text{sum} = i)$.
Let \((p_1, \ldots, p_6)\) and \((q_1, \ldots, q_6)\) be dice. Assume they yield FAIR SUMS, all sums have prob \(1/11\). Then

\[(p_1x + \cdots + p_6x^6)(q_1x + \cdots + q_6x^6) = (1/11)(x^2 + x^3 + \cdots + x^{12})\]

So

\[(p_1 + \cdots + p_6x^5)(q_1 + \cdots + q_6x^5) = (1/11)(1 + x + x^2 + \cdots + x^{10})\]
Two Polynomials

Recap If \((p_1, \ldots, p_6)\) and \((q_1, \ldots, q_6)\) are two loaded dice that yield fair sums then:

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(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5) = (1/11)(1 + x + x^2 + \cdots + x^{10})
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Two Polynomials

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(1 + x + \cdots + x^{10}) = \frac{x^{11} - 1}{x - 1} \text{ hence}
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Recap If \((p_1, \ldots, p_6)\) and \((q_1, \ldots, q_6)\) are two loaded dice that yield fair sums then:

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\[(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5) = \frac{1}{11} \cdot \frac{x^{11} - 1}{x - 1} \text{ hence} \]

\[11(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5)(x - 1) = x^{11} - 1\]
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Real Roots of Left Polynomials

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\( p_1 + \cdots + p_6x^5 \): odd degree, real coefficients, so has \( \geq 1 \) real root.
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Upshot: The left poly has \(\geq 3\) real roots.
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\((x - 1)\) has 1 real root.
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\[ p_1 + \cdots + p_6x^5: \text{ odd degree, real coefficients, so has } \geq 1 \text{ real root.} \]

\[ q_1 + \cdots + q_6x^5: \text{ odd degree, real coefficients, so has } \geq 1 \text{ real root.} \]

\[ (x - 1) \text{ has } 1 \text{ real root.} \]

**Upshot** The Left poly has \( \geq 3 \) real roots.
Real Roots of Right Polynomials (II)

\[11(p_1 + \cdots + p_6x^5)(q_1 + \cdots + q_6x^5)(x - 1) = x^{11} - 1\]

Recall Upshot
The Left poly has \(\geq 3\) real roots.

Let's look at the roots of the right poly:

\[x^{11} - 1 = 0\]

\[x^{11} = 1\]

All roots on complex unit circle. Hence \(\leq 2\) real roots.

Upshot
The Right poly has \(\leq 2\) real roots.

Final Upshot
The left and right poly DIFFER on the number of real roots, so they cannot be the same. Contradiction!
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**Recall Upshot**  The Left poly has $\geq 3$ real roots.
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**Upshot** The Right poly has $\leq 2$ real roots.
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