START RECORDING
Discrete Probability

CMSC 250
Axiomatic Definitions, Basic Problems with Cards
Informal Definition of Probability

• Probability that blah happens:

\[
\frac{\text{# possibilities that }}{\text{# all possibilities}} \text{ blah happens}
\]
**Informal** Definition of Probability

• Probability that *blah* happens:

\[
\frac{\text{# possibilities that *blah* happens}}{\text{# all possibilities}}
\]

• This definition is owed to [Andrey Kolmogorov](https://en.wikipedia.org/wiki/Andrey_Kolmogorov), and assumes *that all possibilities are equally likely*!
First Examples

• Experiment #1: Tossing the same coin 3 times.
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  • What is the probability that I don’t get any heads?
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  • What is the probability that I don’t get any heads?
  • Why?
    • Set of different events?
      • \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} (8 of them)
    • Set of events with no heads:
      • \{TTT\} (1 of them)
    • Hence the answer: \frac{1}{8}
First Examples

• Experiment #1: Tossing the same coin 3 times.
  • What is the probability that I don’t get any heads?
  • Why?
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      • \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} (8 of them)
    • Set of events with no heads:
      • \{TTT\} (1 of them)
    • Hence the answer: \(\frac{1}{8}\)

Implicit assumption: all individual outcomes (HHH, HHT, HTH, ....) are considered equally likely (probability 1/8)
Practice

• Experiment #2: I roll two dice.
  • Probability that I hit seven = ?
    
    \[
    \begin{array}{ccc}
    \frac{1}{12} & \frac{1}{6} & \frac{7}{12} \\
    \text{Something else} & \\
    \end{array}
    \]
• Experiment #2: I roll two dice.
  • Probability that I hit seven = ?
  • Why?
    • Set of different events?
      • \{(1, 1), (1, 2), \ldots , (6, 1)\} (36 of them)
    • Set of events where we hit 7.
      • \{(2, 5), (5, 2), (3, 4), (4, 3), (1, 6), (6, 1)\} (6 of them)
    • Hence the answer: \(\frac{6}{36} = \frac{1}{6}\)
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Practice

• Experiment #2: I roll two dice.
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    • Hence the answer: \(\frac{6}{36} = \frac{1}{6}\)
  • Probability that I hit two = ?
    • Same procedure
      • \(\frac{1}{12}\)
Poker Practice

- Full deck = 52 cards, 13 of each suit:
Poker Practice

• Full deck = 52 cards, 13 of each suit:
  • Flush: 5 cards of the same suit
  • What is the probability of getting a flush?
Probability of a Flush

• How many 5-card hands are there?
Probability of a Flush

• How many 5-card hands are there? $\binom{52}{5}$
Probability of a Flush

• How many 5-card hands are there? \( \binom{52}{5} \)

• How many 5-card hands are flushes?
Probability of a Flush

• How many 5-card hands are there? \( \binom{52}{5} \)
• How many 5-card hands are flushes?
  • Choose a suit in one of 4 ways...
Probability of a Flush

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• How many 5-card hands are flushes?
  • Choose a suit in one of 4 ways...
  • Given suit choose any 5 cards out of 13...
Probability of a Flush

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• How many 5-card hands are flushes?
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  • Given suit choose any 5 cards out of 13...
  • So \( 4 \times \binom{13}{5} \)
Probability of a Flush

• How many 5-card hands are there? \(^\binom{52}{5}\)

• How many 5-card hands are flushes?
  • Choose a suit in one of 4 ways...
  • Given suit choose any 5 cards out of 13...
  • So \(4 \times \binom{13}{5}\)

• So, probability of being dealt a flush is

\[
\frac{4 \times \binom{13}{5}}{\binom{52}{5}}
\]
Probability of a Flush

• Probability of being dealt a flush is

\[
4 \times \frac{\binom{13}{5}}{\binom{52}{5}}
\]
Probability of a Flush

• Probability of being dealt a flush is

\[
4 \times \binom{13}{5} \div \binom{52}{5}
\]

• How likely is this?
Probability of a Flush

• Probability of being dealt a flush is

\[
\frac{4 \times \binom{13}{5}}{\binom{52}{5}}
\]

• How likely is this?
  • Not at all likely: \( \approx 0.002 = 0.2\% \) 😞
Likelihood of a Straight

• Straights are 5 cards of *consecutive rank*
  • Ace can be *either end* (high or low)
  • *No wrap-arounds* (e.g. Q K A 2 3, suits don’t matter)
Likelihood of a Straight

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- What is the probability that we are dealt a straight?
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• What is the probability that we are dealt a straight?

• As before, #possible 5-card hands = \( \binom{52}{5} \)
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• What is the probability that we are dealt a straight?

• As before, #possible 5-card hands = \( \binom{52}{5} \)

• To find out the #straights:
  • Pick lower rank in 10 ways (A-10)
  • Pick a suit in 4 ways
  • Pick the 4 subsequent cards *from any suit* in \( 4^4 \) ways
Likelihood of a Straight

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  • Pick the 4 subsequent cards from any suit in \( 4^4 \) ways

That’s 10 * \( 4^5 \) ways.
So, probability of a straight = \( \frac{10 \times 4^5}{\binom{52}{5}} \)
Caveat on Flushes

• Wikipedia says we’re wrong about flushes!
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- Wikipedia says we’re wrong about flushes!
- Formally, our flushes included (for example) 3h 4h 5h 6h 7h
  - Hands like these are called straight flushes and Wikipedia does not include them.
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  • Hands like these are called straight flushes and Wikipedia does not include them.
  • How many straight flushes are there?
Caveat on Flushes

• Wikipedia says we’re wrong about flushes!

• Formally, our flushes included (for example) 3h 4h 5h 6h 7h
  • Hands like these are called straight flushes and Wikipedia does not include them.
  • How many straight flushes are there?
  • 40. Here’s why:
    • Pick rank: A through 10 (higher ranks don’t allow straights) in 10 ways
    • Pick suit in 4 ways
Probability of Non-Straight Flush...

\[
\frac{4 \times \binom{13}{5} - 40}{\binom{52}{5}} = 0.001965
\]

• This is how Wikipedia defines the probability of a flush. 😊
Probability of a Straight Flush...

\[
\frac{\binom{40}{5}}{\binom{52}{5}} = 0.0000138517
\]
Probability of a Straight Flush...

\[
\frac{40}{\binom{52}{5}} = 0.0000138517
\]

The expected # hands you need to play to get a straight flush is then

\[
\left\lceil \frac{1}{0.0000138517} \right\rceil = 72,194
\]
Same Caveat for Straights

• From the #straights we computed we will have to subtract the 40 possible straight flushes to get...

\[
\frac{10 \times 4^5 - 40}{\binom{52}{5}} = 0.003925
\]
Same Caveat

• From the #straights we computed we will have to subtract the 40 possible straight flushes to get...

$$\frac{10 \times 4^5 - 40}{52 \choose 5} = 0.003925 > 0.001965 = \text{probability of flush}$$

• *Flushes, being more rare, beat straights in poker.*
Probability of a Pair

• Try to calculate the probability of a pair!
Try to calculate the probability of a pair!

Perhaps you thought of the problem like this:

- The denominator will be $\binom{52}{5}$ (easy), so let’s focus on the numerator:
  1. First choose rank in 13 ways.
  2. Then, choose two of four suits in $\binom{4}{2} = 6$ ways.
  3. Then, choose 3 cards out of 50 in $\binom{50}{3}$ ways.
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Numerator: $13 \times 6 \times \binom{50}{3}$
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\[\text{Numerator: } 13 \times 6 \times \binom{50}{3}\]

• So, probability = \[\frac{13 \times 6 \times \binom{50}{3}}{\binom{52}{5}}\]
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• So, probability = \[
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Is this accurate?

Yes  No
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  • So, probability = \( \frac{13 \times 6 \times \binom{50}{3}}{\binom{52}{5}} \)

Is this accurate?

Severe overcount!

Yes  No
Don’t Count Better Hands!

• In the computation before, we included:
  • 2-of-a-kind
  • 3-of-a-kind
  • 4-of-a-kind
  • Full House
Don’t Count Better Hands!

• In the computation before, we included:
  • 2-of-a-kind
  • 3-of-a-kind
  • 4-of-a-kind
  • Full House

• To properly compute, we would have to subtract all better hands possible with at least one pair.
Joint Probability
Joint Probability ("AND" of Two Events)

• The probability that two events A and B occur simultaneously is known as the joint probability of A and B and is denoted in a number of ways:
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  - $P(A \cap B)$ (Most useful from a set-theoretic perspective; we’ll be using this)
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  • $P(A \cap B)$ (Most useful from a set-theoretic perspective; we’ll be using this)
  • $P(A, B)$ (One sees this a lot in Physics books)
Joint Probability ("AND" of Two Events)

- The probability that two events A and B occur simultaneously is known as the joint probability of A and B and is denoted in a number of ways:
  - \( P(A \cap B) \) (Most useful from a set-theoretic perspective; we’ll be using this)
  - \( P(A, B) \) (One sees this a lot in Physics books)
  - \( P(AB) \) (Perhaps most convenient, therefore most common)
Calculating Joints

• Probability that the first coin toss is heads and the second coin toss is tails
Calculating Joints

• Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$
Calculating Joints

• Probability that the first coin toss is heads and the second coin toss is tails \( \frac{1}{2} \times \frac{1}{2} \)
• Probability that the first die is at most a 2 and the second one is 5 or 6
Calculating Joints

• Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$

• Probability that the first die is at most a 2 and the second one is 5 or 6
  • # outcomes of die roll is 6
  • # outcomes where first die is at most 2 is 2
    • Hence, probability of first die roll being at most 2 is $\frac{1}{3}$
Calculating Joints

- Probability that the first coin toss is heads and the second coin toss is tails: \( \frac{1}{2} \times \frac{1}{2} \)
- Probability that the first die is at most a 2 and the second one is 5 or 6
  - # outcomes of die roll is 6
  - # outcomes where first die is at most 2 is 2
    - Hence, probability of first die roll being at most 2 is \( \frac{1}{3} \)
  - Similarly, probability of second die roll being 5 or 6 is \( \frac{1}{3} \).
- Hence, probability that both events happen (joint probability) is \( \frac{1}{9} \).
Calculating Joints

• Jason’s going to flip a coin and then pick a card from a 52-card deck.
  • Probability that the coin is heads and the card has rank 8?

\[
\frac{1}{2} \quad \frac{1}{26} \quad \frac{1}{32} \quad \text{Something else}
\]
Calculating Joints

• Jason’s going to flip a coin and then pick a card from a 52-card deck
  • Probability that the coin is heads and the card has rank 8?

  \[
  \begin{array}{c}
  \frac{1}{2} \\
  \frac{1}{26} \\
  \frac{1}{32} \\
  \text{Something else}
  \end{array}
  \]

• This is because \( P(\text{coin} = H) = \frac{1}{2} \) and \( P(\text{card_rank} = 8) = \frac{4}{52} = \frac{1}{13} \)
  • So their joint probability is \( \frac{1}{2} \times \frac{1}{13} = \frac{1}{26} \)
The Law of Joint Probability

\[ P(A \cap B) = P(A) \cdot P(B) \]

\[ P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i) \]
The Law of Joint Probability

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- Unfortunately, this “law” is not always applicable!
The Law of Joint Probability

\[ P(A \cap B) = P(A) \cdot P(B) \]
\[ P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i) \]

- Unfortunately, this “law” is not always applicable!
- It is applicable only when all the different events \( A_i \) are \textit{independent} (sometimes called \textit{marginally independent}) of each other.
- Let’s look at an example.
What If The Events Influence Each Other?

• Probability that a die is even and that it is 2.
What If The Events Influence Each Other?

- Probability that a die is even and that it is 2.
  - Probability that the die is even $= \frac{1}{2}$
What If The Events Influence Each Other?

• Probability that a die is even and that it is 2.
  • Probability that the die is even = \( \frac{1}{2} \)
  • Probability that the die is two = \( \frac{1}{6} \)
What If The Events Influence Each Other?

• Probability that a die is even and that it is 2.
  • Probability that the die is even = \( \frac{1}{2} \)
  • Probability that the die is two = \( \frac{1}{6} \)
  • Probability the die is even \textbf{and} the die is two = \( \frac{1}{12} \) ???
What If The Events Influence Each Other?

• Probability that a die is even and that it is 2.
  • Probability that the die is even = \( \frac{1}{2} \)
  • Probability that the die is two = \( \frac{1}{6} \)
  • Probability the die is even and the die is two = \( \frac{1}{12} \) ???
  • NO!
    • What is the probability that the die is even and the die is 2?
What If The Events Influence Each Other?

• Probability that a die is even and that it is 2.
  • Probability that the die is even = \( \frac{1}{2} \)
  • Probability that the die is two = \( \frac{1}{6} \)
  • Probability the die is even and the die is two = \( \frac{1}{12} \) ???
  • NO!
    • What is the probability that the die is even and the die is 2?
Set-Theoretic Interpretation

• Notice that the event A: “Die roll is even” is a superset of the event B: “Die roll comes 2”
Set-Theoretic Interpretation

• Notice that the event A: “Die roll is even” is a superset of the event B: “Die roll comes 2”

• Since $A \cap B = A$, $P(A \cap B) = P(A) = \frac{1}{6}$
Calculating Joints

• The University of Southern North Dakota offers a Discrete Mathematics Course where the possible grades are A through G. (No + or -)
Calculating Joints

• The University of Southern North Dakota offers a Discrete Mathematics Course where the possible grades are A through G. (No + or -)
• What is the probability that Jason gets both an A and a G in that course?
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• What is the probability that Jason gets both an A and a G in that course?
  • Clearly, it can’t be

\[
(\text{probability Jason gets an A}) \times (\text{probability Jason gets a B}) = \frac{1}{7} \times \frac{1}{7} = \frac{1}{49}
\]
Calculating Joints

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- What is the probability that Jason gets both an A and a G in that course?
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\frac{1}{7} \times \frac{1}{7} = \frac{1}{49}
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Calculating Joints

• **The University of Southern North Dakota** offers a Discrete Mathematics Course where the possible grades are A through G. (No + or -)

• What is the probability that Jason gets both an A and a G in that course?
  • Clearly, it *can’t* be
    \[
    \text{(probability Jason gets an A)} \times \text{(probability Jason gets a B)} = \frac{1}{7} \times \frac{1}{7} = \frac{1}{49}
    \]

• It is 0. Those two events cannot happen *jointly*!
Calculating Joints

- The University of Southern North Dakota offers a Discrete Mathematics Course where the possible grades are A through G. (No + or -)
- What is the probability that Jason gets both an A and a G in that course?
  - Clearly, it can’t be 
    \[ \frac{1}{7} \times \frac{1}{7} = \frac{1}{49} \]
  - It is 0. Those two events cannot happen \textit{jointly}!
  - Events such as these are called disjoint or mutually disjoint.
Set-Theoretic Interpretation

• $A = \text{“Jason gets an A in USND’s 250”}$
• $G=\text{“Jason gets a G in USND’s 250”}$
Set-Theoretic Interpretation

• A = “Jason gets an A in USND’s 250”
• G=“Jason gets a G in USND’s 250”

Note that $A \cap G = \emptyset$, so there are no common outcomes.

• So $P(A \cap G) = 0$
Calculating Joints

• I have my original die again.
  • Probability that it comes up 1, 2 or 3 = \( \frac{1}{2} \)
  • Probability that it comes up 3, 4 or 5 = \( \frac{1}{2} \)
  • What is the probability that it comes up 1, 2 or 3 and 3, 4 or 5?
Calculating Joints

• I have my original die again.
  • Probability that it comes up 1, 2 or 3 = $\frac{1}{2}$
  • Probability that it comes up 3, 4 or 5 = $\frac{1}{2}$
  • What is the probability that it comes up 1, 2 or 3 and 3, 4 or 5?
Calculating Joints

• I have my original die again.
  • Probability that it comes up 1, 2 or 3 = $\frac{1}{2}$
  • Probability that it comes up 3, 4 or 5 = $\frac{1}{2}$
  • What is the probability that it comes up 1, 2 or 3 and 3, 4 or 5?

Note that the only common outcome between the two events is 3, which can come up only once out of six possibilities.
Set-Theoretic Interpretation

- Let $A = \text{dice comes up 1, 2, or 3}$
- Let $B = \text{dice comes up 3, 4, or 5}$
- Let $C = \text{dice comes up 1, 2, 3, 4, 5 OR 6}$
Set-Theoretic Interpretation

• Let A = dice comes up 1, 2, or 3
• Let B = dice comes up 3, 4, or 5
• Let C = dice comes up 1, 2, 3, 4, 5 OR 6

• Then, probability that the dice comes up 3 = $\frac{1}{6}$
Dependent and Independent Events
Independent Events \textit{(informally)}

• Two events are independent if \textcolor{red}{one does not influence the other.}
Independent Events *(informally)*

• Two events are independent if **one does not influence the other.**

• Examples:
  • The event E1 = “first coin toss” and E2 = “second coin toss”
  • With the same die, the events E1 = “roll 1”, E2 = “roll 2”, E3 = “roll 3”
  • Jason flips a coin and then picks a card.
Independent Events (informally)

• Two events are independent if one does not influence the other.

• Examples:
  • The event $E_1 = \text{“first coin toss”}$ and $E_2 = \text{“second coin toss”}$
  • With the same die, the events $E_1 = \text{“roll 1”}$, $E_2 = \text{“roll 2”}$, $E_3 = \text{“roll 3”}$
  • Jason flips a coin and then picks a card.

• Counter-examples:
  • $E_1 = \text{“Die is even”}$, $E_2=\text{“Die is 6”}$
  • $E_1= \text{“Grade in 250”}$ and $\text{“Passing 250”}$
Law of Joint Probability (*informally*)

- Two events are independent if *one does not influence the other.*
  - This definition is a but *too informal*, so mathematicians tend to avoid it.
Law of Joint Probability \textit{(informally)}

• Two events are independent if \textcolor{orange}{one does not influence the other.}
  • This definition is a but \textcolor{orange}{too informal}, so mathematicians tend to avoid it.
• Formally, we define that $A$ and $B$ are \textcolor{blue}{independent} if

\[
P(A \cap B) = P(A) \cdot P(B)
\]
Disjoint or Independent?

1. $E_1 = \text{“It rains in College Park, MD today”}$
   $E_2 = \text{“It rains in Athens, Greece today”}$

Disjoint  Independent  Both  Neither
Disjoint or Independent?

1. $E_1 = \text{"It rains in College Park, MD today"}$
   $E_2 = \text{"It rains in Athens, Greece today"}$

Disjoint Independent Both Neither

Weather is weird!
1. \( E_1 = \) “It rains in College Park, MD today”
   \( E_2 = \) “It rains in Athens, Greece today”

2. \( E_1 = \) “It rains in College Park, MD today”
   \( E_2 = \) “It is sunny in College Park, MD today”

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Disjoint or Independent?

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   $E_2 = \text{“It rains in Athens, Greece today”}$
   - Disjoint
   - Independent
   - Both
   - Neither

2. $E_1 = \text{“It rains in College Park, MD today”}$
   $E_2 = \text{“It is sunny in College Park, MD today”}$
   - Disjoint
   - Independent
   - Both
   - Neither

3. $E_1 = \text{Die #1 comes at most 4}$
   $E_2 = \text{Die #2 comes at least 5}$
   - Disjoint
   - Independent
   - Both
   - Neither

Weather is weird!
1. $E_1 = \text{“It rains in College Park, MD today”}$  
   $E_2 = \text{“It rains in Athens, Greece today”}$

2. $E_1 = \text{“It rains in College Park, MD today”}$  
   $E_2 = \text{“It is sunny in College Park, MD today”}$

3. $E_1 = \text{Die #1 comes at most 4}$  
   $E_2 = \text{Die #2 comes at least 5}$
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4. $E_1 = \text{Student gets a C}$ 
   $E_2 = \text{Student passes the class}$

**Disjoint or Independent?**

- Weather is weird!
1. $E_1 = “It rains in College Park, MD today” 
   $E_2 = “It rains in Athens, Greece today”$

2. $E_1 = “It rains in College Park, MD today” 
   $E_2 = “It is sunny in College Park, MD today”$

3. $E_1 = Die #1 comes at most 4$
   $E_2 = Die #2 comes at least 5$

4. $E_1 = Student gets a C$
   $E_2 = Student passes the class$

Weather is weird!
Recap: “Disjoint” vs “Independent”

- Friends don’t let friends get confused between “disjoint” and “independent”!

<table>
<thead>
<tr>
<th>Disjoint</th>
<th>Independent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Has a set-theoretic interpretation!</td>
<td>Has a causality interpretation!</td>
</tr>
<tr>
<td>Means that $P(A \cap B) = 0$</td>
<td>Means that $P(A \cap B) = P(A) \cdot P(B)$</td>
</tr>
<tr>
<td>Means that $P(A \cup B) = P(A) + P(B)$</td>
<td>Means that $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$</td>
</tr>
</tbody>
</table>
Disjoint Probability ("OR" of Two Events)

• Jason rolls two dice.
  • What is the probability that he rolls a 7 or a 9?
Disjoint Probability ("OR" of Two Events)

• Jason rolls two dice.
  • What is the probability that he rolls a 7 or a 9?
  • #Ways to roll a 7 is 6.
  • #Ways to roll a 9 is 4: (6, 3), (5, 4), (4, 5), (3, 6)
  • #Ways to roll a 7 OR a 9 is then 10.
  • Therefore, the probability is \( \frac{10}{36} = \frac{5}{18} \)
  • Key: Rolling a 7 and a 9 are disjoint events.
Disjoint Probability (‘‘OR’’)

• 52-card deck
• Probability of drawing a face card (J, Q, K) or a heart
Disjoint Probability ("OR")

• 52-card deck
• Probability of drawing a face card (J, Q, K) or a heart
  • Are these disjoint?
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- 52-card deck
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  - Are these disjoint?
    - NO, for example, Queen of hearts
- How big is $\text{Face Card} \cup \text{Hearts}$?
Disjoint Probability (“OR”)

• 52-card deck
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  • NO, for example, Queen of hearts
• How big is \( \text{Face}\_\text{Card} \cup \text{Hearts} \) (abbrv. \( F, H \) below)?
  • Use law of inclusion / exclusion!

\[
|F \cup H| = |F| + |H| - |F \cap H| = 12 + 13 - 3 = 22
\]
Disjoint Probability ("OR")

• 52-card deck
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$$|F \cup H| = |F| + |H| - |F \cap H| = 12 + 13 - 3 = 22$$

• So probability $= \frac{22}{52} = \frac{11}{26}$. 
Alternative Viewpoint

- \( P(F) = \frac{12}{52} \)
- \( P(H) = \frac{13}{52} \)
- \( P(F \cap H) = \frac{3}{52} \)
- \( P(F \cup H) = P(F) + P(H) - P(F \cap H) \)
Alternative Viewpoint

- \( P(F) = \frac{12}{52} \)
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- \( P(F \cap H) = \frac{3}{52} \)
- \( P(F \cup H) = P(F) + P(H) - P(F \cap H) \)

- We can also do:

\[
\frac{\binom{13}{4}(\binom{4}{1}\binom{4}{2}) \times 4^3}{\binom{52}{5}}
\]
Probability of Unions

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
Probability of Unions

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

- If A and B are independent, we have

\[ P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B) \]
Probability of Unions

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• If A and B are disjoint, we have

\[ P(A \cup B) = P(A) + P(B) \]
Probability of Unions of 3 Sets

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) \]
\[ - P(A \cap B) - P(B \cap C) - P(A \cap C) \]
\[ + P(A \cap B \cap C) \]
Probability of Unions of 3 Sets

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \]

• If A, B and C are pairwise independent, we have:

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A) \cdot P(B) - P(B) \cdot P(C) - P(A) \cdot P(C) + P(A \cdot B \cdot C) \]
Probability of Unions of 3 Sets

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) \]

\[ - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \]

• If A, B and C are **pairwise independent**, we have:

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A) \cdot P(B) - P(B) \cdot P(C) - P(A) \cdot P(C) + P(A \cdot B \cdot C) \]

• If A, B and C are **pairwise disjoint** (so \( A \cap B = A \cap C = B \cap C = \emptyset \), so clearly \( A \cap B \cap C = \emptyset \)), we have

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) \]
Conditional Probability and Bayes’ Law
Conditional Probability

- If A occurs, then is B
  a) More likely?
  b) Equally likely?
  c) Less likely?
Conditional Probability

• If A occurs, then is B
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• Any of these could happen, it depends on the relationship between A and B.
Conditional Probability

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Examples

• We roll two dice
  • Event $A = \text{“Sum of the dice } S \equiv 0 \ (mod\ 4)\text{”}$
    • Note that $P(A) = \frac{9}{36} = \frac{1}{4}$, since we have nine rolls of the dice that sum to a multiple of 4:
      $(1, 3), (2, 2), (3, 1), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (6, 6)$
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    • Note that $P(B) = \frac{6}{36} = \frac{1}{6}$
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• What is the probability of A given B?
Examples

• What is the probability of A given B?
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  • Outcomes of A are (1, 3), (2, 2), (3, 1), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (6, 6)
  • Outcomes of B are (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
  • Outcomes of rolling two dice: (1, 1), (1, 2), ..., (6, 5), (6, 6)
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• As discussed, \( P(A) = \frac{9}{36} = \frac{1}{4} \)
• However, once B occurs, instead of 36 outcomes, we now have...
Examples

• What is the probability of $A$ given $B$?
  
  • Outcomes of $A$ are (1, 3), (2, 2), (3, 1), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (6, 6)
  
  • Outcomes of $B$ are (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
  
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  • Only 2 of them are outcomes that correspond to A.
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  • Outcomes of rolling two dice: (1, 1), (1, 2), ...., (6, 5), (6, 6)

• As discussed, \( P(A) = \frac{9}{36} = \frac{1}{4} \)

• However, once B occurs, instead of 36 outcomes, we now have... 6 outcomes.
  • Only 2 of them are outcomes that correspond to A.
  • Therefore, the probability of A given B is \( \frac{2}{6} = \frac{1}{3} \)
Examples

- We once again two roll dice
  - Event A = “Sum of the dice is $\geq 8$”
  - Event B = “First die is 4”
Examples

• We once again two roll dice
  • Event A = “Sum of the dice is ≥ 8”
  • Event B = “First die is 4”

• If B happens, what is your intuition about the probability of A?
Examples

• We once again two roll dice
  • Event A = “Sum of the dice is \( \geq 8 \)”
  • Event B = “First die is 4”

• If B happens, what is your intuition about the probability of A?

Go up  Go down  Stay the same  Unknown to science
Examples

• We once again two roll dice
  • Event A = “Sum of the dice is $\geq 8$”
  • Event B = “First die is 4”

• If B happens, what is your intuition about the probability of A?

Let’s see if your intuition was correct!
Examples

• We once again two roll dice
  • Event A = “Sum of the dice is $\geq 8$” $P(A) = ?$ (work on it)
  • Event B = “First die is 4”
Examples

• We once again two roll dice
  • Event A = “Sum of the dice is $\geq 8$” $P(A) = \frac{15}{36} = \frac{5}{12}$
  • Event B = “First die is a 4”
Examples

• We once again roll two dice
  • Event A = “Sum of the dice is ≥ 8” \( P(A) = \frac{15}{36} = \frac{5}{12} \)
  • Event B = “First die is a 4” \( P(B) = \frac{1}{6} \)
Examples

• We once again two roll dice
  • Event A = “Sum of the dice is ≥ 8” \( P(A) = \frac{15}{36} = \frac{5}{12} \)
  • Event B = “First die is a 4” \( P(B) = \frac{1}{6} \)
  • Prob of A given B = Prob second dice is 4, 5, or 6 = \( \frac{3}{6} = \frac{1}{2} > \frac{5}{12} \)

By just \( \frac{1}{12} \)...
Conditional Probability

• Let $A, B$ be two events. The conditional probability of $A$ given $B$, denoted $P(A \mid B)$, is defined as follows:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$
Re-Thinking Independent Events

• **Alternative definition of independent events**: Two events A and B will be called marginally independent, or just independent for short, if and only if

\[ P(A|B) = P(A) \]
Re-Thinking Independent Events

• **Alternative definition of independent events**: Two events $A$ and $B$ will be called marginally independent, or just independent for short, if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$


• Applying the definition of $P(A|B)$ we have:

  • $P(A \cap B) = P(A) \Rightarrow P(A \cap B) = P(A) \cdot P(B)$, which is a relationship we had reached earlier when discussing the joint probability.
Complex Probabilities

• Suppose that I have two dice: a six-sided one and a ten-sided one.
Complex Probabilities

• Suppose that I have two dice: a six-sided one and a ten-sided one.

• I pick either one of them with probability $\frac{1}{2}$ and roll it.
  • What’s the probability that the die comes up 6? (work on this yourselves NOW)
Complex Probabilities

• Suppose that I have two dice: a six-sided one and a ten-sided one.
  • I pick either one of them with probability $\frac{1}{2}$
    • What’s the probability that the die comes up 6? (work on this yourselves NOW)

\[
P(Roll = 6) = P(Roll = 6, Die = 6) + P(Roll = 6, Die = 10) = \\
= P(Roll = 6|Die = 6) \times P(Die = 6) + P(Roll = 6|Die = 10) \times P(Die = 10) \\
= \frac{1}{6} \times \frac{1}{2} + \frac{1}{10} \times \frac{1}{2} = \frac{1}{12} + \frac{1}{20} = \frac{2}{15} \approx 0.1333 ...\]
Complex Probabilities

• Suppose that I have two dice: a six-sided one and a ten-sided one.
Complex Probabilities

• Suppose that I have two dice: a six-sided one and a ten-sided one.
• Now we change the problem so that we pick the ten-sided die with prob $\frac{5}{9}$ and the six-sided die with prob $\frac{4}{9}$. 
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• Intuitively, will the probability that I come up with a 6...
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• Suppose that I have two dice: a six-sided one and a ten-sided one.
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Go up  Go down  Stay the same
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Let’s see if your intuition was correct!
Complex Probabilities

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• Now we change the problem so that we pick the ten-sided die with prob $\frac{5}{9}$ and the six-sided die with prob $\frac{4}{9}$.

• What’s the probability that I come up with a 6?

$$P(\text{Roll} = 6) = P(\text{Roll} = 6, \text{Die} = 6) + P(\text{Roll} = 6, \text{Die} = 10) =$$

$$= P(\text{Roll} = 6|\text{Die} = 6) \times P(\text{Die} = 6) + P(\text{Roll} = 6, \text{Die} = 10) \times P(\text{Die} = 10) =$$

$$= \frac{1}{6} \times \frac{4}{9} + \frac{1}{10} \times \frac{5}{9} = \frac{2}{27} + \frac{1}{18} = \frac{7}{54} \approx 0.130 < 0.133$$
Bayes’ Law

• Suppose $A$ and $B$ are events in a sample space $\Omega$. Then, the following is an identity:

$$P(A|B) = P(B|A) \frac{P(A)}{P(B)}$$

known as **Bayes’ Law**
Questions

• If $P(A|B) = P(A)$, is it the case that $P(B|A) = P(B)$?

Yes  No
Questions

• If $P(A|B) = P(A)$, is it the case that $P(B|A) = P(B)$?

  - Yes
  - No

• Substituting $P(A|B)$ with $P(A)$ in the formulation of Bayes’ Law, we have:

\[
P(A) = P(B | A) \cdot \frac{P(A)}{P(B)} \Rightarrow 1 = \frac{P(B|A)}{P(B)} \Rightarrow P(B|A) = P(B)
\]
Questions

• If $P(A|B) = P(A)$, is it the case that $P(B|A) = P(B)$?

  \[ P(A) = P(B | A) \cdot \frac{P(A)}{P(B)} \Rightarrow 1 = \frac{P(B|A)}{P(B)} \Rightarrow P(B|A) = P(B) \]

  Yes \hspace{1cm} No

(A ind B) iff (B ind A)

• Substituting $P(A|B)$ with $P(A)$ in the formulation of Bayes’ Law, we have:
Questions

• If $P(B) = 0$, then is $P(A|B)$ also 0?

Yes

No
Questions

• If \( P(B) = 0 \), then is \( P(A|B) \) also 0?

  
  Yes  No

  

• It is **undefined**, since \( P(A \mid B) = P(B \mid A) \cdot \frac{P(A)}{P(B)} \)
STOP
RECORDING