Homework 2, MORALLY Due Feb 12

1. (28 points-7 points each)

   (a) Consider the formula

   $$(x_{11} \vee x_{12}) \wedge (x_{21} \vee x_{22}) \wedge (x_{31} \vee x_{32}) \wedge (x_{41} \vee x_{42}).$$

   How many satisfying assignments does this formula have? Justify!

   SOL TO 1a

   Each clause can be satisfied by either TF or FT or TT. There is not interaction between the clauses so these are all independent of each other. Hence there are 3 options for each clause. Hence the number of satisfying assignments is

   $$3 \times 3 \times 3 \times 3 = 81.$$

   END OF SOL TO 1a

   (b) Let $n \in \mathsf{N}$ with $n \geq 3$. Consider the formula

   $$(x_{11} \vee x_{12}) \wedge (x_{21} \vee x_{22}) \wedge \cdots \wedge (x_{n1} \vee x_{n2}).$$

   How many satisfying assignments does this formula have? Justify! (Note that it may be a function of n.)

   SOL TO 1b

   Each clause can be satisfied by either TF or FT or TT. There is not interaction between the clauses so these are all independent of each other. Hence there are 3 options for each clause. Hence the number of satisfying assignments is

   $$3 \times 3 \times \cdots \times 3 = 3^n.$$

   END OF SOL TO 1b

   (c) Consider the formula

   $$(x_1 \vee \neg x_2) \wedge (x_2 \vee \neg x_3) \wedge (x_3 \vee \neg x_4) \wedge (x_4 \vee \neg x_1).$$

   How many satisfying assignments does this formula have? Justify! (Do not use a truth table.)

SOL TO 1c

We can easily see that $(x_1, x_2, x_3, x_4) = (T, T, T, T)$ is a satisfying assignment. Are there any others? Note that all of the variables are symmetric so if we show that x_1 cannot be F, then that will show that no variable can be false.

If $x_1 = F$ then $x_2 = F$ by Clause 1.

If $x_2 = F$ then $x_3 = F$ by Clause 2.

If $x_3 = F$ then $x_4 = F$ by Clause 3.

If $x_4 = F$ then $x_1 = F$ by Clause 4.

AH- so if $x_1 = F$ then $x_2 = x_3 = x + 4 = F$.

So there are two satisfying assignment.

END OF SOL TO 1c

(d) Give an example of a 2CNF formula which uses 4 variables and is NOT satisfiable.

SOL TO 1d

We use w, x, y, z.

We will make any combination of w and x not work and then add in y, z.

$$(w \vee x) \wedge (\neg w \vee x) \wedge (w \vee \neg x) \wedge (\neg w \vee \neg x) \wedge (y \vee z).$$

END OF SOL TO 1d

2. (20 points) Let $n \in \mathsf{N}$. Two formulas ϕ_1 and ϕ_2 are n-*Equiv* if their truth tables DIFFER on exactly n rows. Note that logical equivalence is 0-Equiv.

   (a) (10 points) Give an example of two formulas ϕ_1, ϕ_2, each on 3 variables, that are 1-Equiv. (Hint: This is easier if you make them in DNF form.)

   SOL to 2a

   $\phi_1(x, y, z) = (x \wedge y \wedge z)$ This only has one satisfying assignment (T, T, T).

   $\phi_2(x, y, z) = (x \wedge y \wedge z) \vee (x \wedge y \wedge \neg z)$ This only has two satisfying assignment (T, T, T) and (T, T, F).

   Hence these two agree on all rows except (T, T, F).

   END OF SOL 2a

   (b) (10 points) Give an example of three formulas ϕ_1, ϕ_2, ϕ_3, each on 3 variables, such that the following hold:

      i. ϕ_1 and ϕ_2 are 1-Equiv.
      ii. ϕ_2 and ϕ_3 are 1-Equiv.
      iii. ϕ_1 and ϕ_3 ar 2-Equiv.

   SOL to 2b

   $\phi_1(x, y, z) = (x \wedge y \wedge z)$. This has one satisfying assignment (T, T, T).

   $\phi_2(x, y, z) = (x \wedge y \wedge z) \vee (x \wedge y \wedge \neg z)$. This has two satisfying assignment (T, T, T) and (T, T, F).

   $\phi_3(x, y, z) = (x \wedge y \wedge \neg z)$. This has one satisfying assignment (T, T, F).

   You can check that ϕ_1, ϕ_2, ϕ_3 satisfy the conditions we wanted.

   END OF SOL to 2b

3. (20 points – 5 points each) For each of the following statements write the negation without using any negations signs.

   (a) $x \neq 4$

   SOL to 3a

   $x = 4$.

   END OF SOL 3a

   (b) $(x_1 \leq x_2) \wedge (x_1 \leq x_3)$

   SOL to 3b

   $(x_1 > x_2) \vee (x_1 > x_3)$.

   END OF SOL 3b

   (c) $(x \leq 5) \vee (x \geq 15)$

   SOL to 3c

   $(x > 5) \wedge (x_1 < x_3)$.

   END OF SOL 3c

   (d) $x < y < z$

   SOL to 3d

   $(x \geq y) \vee (y \geq z)$

   END OF SOL 3d

4. (32 points-8 points each) You are designing an algorithm for CNFSAT. I will incompletely describe some short cuts you can take. Fill in the BLANK

   The input is of the form

   $C_1 \wedge \cdots \wedge C_m$

   where each C_i is an OR of literals (a literal is a var or its negation).

   (a) If $C_1 = (x_3)$ then you can do $BLANK_1$.

      SOL to 4a

      You can set x_3 to T.

      END OF SOL 4a

   (b) If x_4 appears in the formula but $\neg x_4$ never appears then you can do $BLANK_2$.

      SOL to 4b

      You can set x_4 to T.

      END OF SOL 4b

   (c) If $C_2 = (x_8)$ and $C_3 = (x_9)$ and $C_4 = (\neg x_8 \vee \neg x_9)$ then you can do $BLANK_3$.

      SOL to 4c

      You can say NOT SATISFIED.

      END OF SOL 4c

   (d) If $C_4 = (x_{10} \vee \neg x_{11} \vee x_{12} \vee \neg x_{12})$ then you can do $BLANK_4$.

      SOL to 4d

      Remove C_4 entirely since you know that it is satisfied.

      END OF SOL 4d