Homework 3, MORALLY Due Feb 19

1. (30 points–6 points each) In this problem the domain is N.

   (a) Express the following statements using quantifiers. There exist an (x, y, z), with $x, y, z \ge 2$ all distinct, such that $x^2 + y^2 = z^2$.

   SOL TO 1a

   $(\exists x, y, z)[(x \ge 2)\wedge(y \ge 2)\wedge(z \ge 2)\wedge(x \ne y)\wedge(x \ne z)\wedge(y \ne z)\wedge x^2+y^2 = z^2.]$

   END OF SOL TO 1a

   (b) The statement in 1a is TRUE. Give 5 examples of (x, y, z) that satisfy it.

   SOL TO 1b

   We give an infinite number of (x, y, z).

   $(3, 4, 5)$

   $(6, 8, 10)$

   $(9, 12, 15)$

   More generally, for $k \in \mathsf{N}$, $(3k, 4k, 5k)$

   END OF SOL TO 1b

   (c) Express the following statements using quantifies. There exist an INFINITE NUMBER of (x, y, z), with $x, y, z \ge 2$ and all distinct, such that $x^2 + y^2 = z^2$. (This happens to be TRUE but you do not need that for this problem.)

   SOL TO 1c

   Let $(\exists x, y, z)[\phi(x, y, z)]$ be the formula for problem 1a.

   We need to say that there are an infinite number of (x, y, z).

   If it was just an infinite number of x I would say

   $(\forall w)(\exists x)[x \ge w]$.

   But how do I do this for a triple of numbers?

   If there was an infinite number of triples

   (x_1, y_1, z_1), (x_2, y_2, z_2), $\ldots$

   then there is an infinite subsequence where the SUM increases. SO

   $(\forall w)(\exists x, y, z)[x + y + z \ge w \wedge \phi(x, y, z)]$

   END OF SOL TO 1c

(d) Express the following statements using quantifies. There is NO x, y, z, n with $x, y, z \geq 2$ and $n \geq 3$ such that $x^n + y^n = z^n$.

SOL TO 1d

$(\forall x, y, z, n)$

$$[((x \geq 2) \wedge (y \geq 2) \wedge (z \geq 2) \wedge (n \geq 3) \Rightarrow x^n + y^n \neq z^n.]$$

END OF SOL TO 1d

(e) The statement in Part d is TRUE. It is called *Fermat's Last Theorem*. Look it up and write a paragraph about it including who proved it, when, and how long it was open for.

SOL TO 1e

Solution Omitted.

END OF SOL TO 1e

2. (30 points–15 points each) In this problem the domain is R.

   (a) Express the following statements using quantifies. Every polynomial with real coefficients, of degree 3, has a real solution.

   SOL TO 2a

   $(\forall a_0, a_1, a_2, a_3)(\exists x)[a_3 \neq 0 \Rightarrow a_3x^3 + a_2x^2 + a_1x + a_0 = 0]$

   END OF SOL TO 2a

   (b) The statement in Part a is TRUE! Give a proof (it can be informal, this is NOT Math 410).

   SOL TO 2b

   Let $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

   Assume $a_3 > 0$.

   As $x \Rightarrow \infty$ then $f(x) \Rightarrow \infty$ since a_3x^3 becomes the dominant term.

   As $x \Rightarrow -\infty$ then $f(x) \Rightarrow -\infty$ since a_3x^3 becomes the dominant term.

   Since $f(x)$ takes on both positive and negative values, it must also take on 0.

   Similar for $a_3 < 0$.

   END OF SOL TO 2b

3. (40 points–8 points each)

For this problem we use the following standard terminology:

A domain is *dense* if $(\forall x, y)[x < y \Rightarrow (\exists z)[x < z < y]]$.

A domain has a *min element* if $(\exists x)(\forall y)[x \leq y]$.

A domain has a *max element* if $(\exists x)(\forall y)[y \leq x]$.

In this problem we list conditions on a domain. EITHER give a domain that satisfies the conditions OR state that there is NO such domain (no proof required).

(a) D is finite and dense.

SOL TO 3a

There are two such domains:

$D = \emptyset$ is finite. Its dense vacuously.

$D = \{1\}$ is finite. Its dense vacuously.

END OF SOL TO 3a

(b) D is finite and dense and has ≥ 2 elements.

SOL TO 3b

There are no such domains:

Let $x, y \in D$.

There is an elements z in between x, y so

$x < z < y$

There is an element between x and z, and between z and y.

Etc. Hence there is an infinite number of elements in D.

END OF SOL to 3b

(c) D is infinite and not dense

SOL TO 3c

Z

END OF SOL TO 3c

(d) D is infinite, has a min element, has a max element, and is dense.

SOL TO 3d

$[0, 1] \cap \mathsf{Q}$ or $[0, 1] \cap \mathsf{R}$.

END OF SOL TO 3d

(e) D is infinite, has a min element, has a max element, and is NOT dense.

SOL TO 3e

$$\{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}.$$

END OF SOL TO 3e