More Combinatorics and Probability Problems CMSC 250

1 Counting

1. How many different three-letter initials can be made where the first initial is A?

$$1 \cdot 26 \cdot 26 = 676$$

2. How many bit strings are there of length six or less, not counting the empty string?

   $6 : 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$
   $5 : 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$
   $4 : 2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$
   $3 : 2 \cdot 2 \cdot 2 = 2^3 = 8$
   $2 : 2 \cdot 2 = 2^2 = 4$
   $1 : 2$

$$64 + 32 + 16 + 8 + 4 + 2 = 126$$

3. How many ways are there to seat six people around a circular table, where two seatings are considered the same when everyone has the same two neighbors without regard to whether they are right or left neighbors?

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2} = \frac{5!}{2} = 60$$

4. How many license plates can be made using either two or three uppercase English letters followed by either two or three digits?

   2 English Letters and 2 Digits: $26 \cdot 26 \cdot 10 \cdot 10 = 67600$
   2 English Letters and 3 Digits: $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 676000$

3 English Letters and 2 Digits: $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 = 1757600$
3 English Letters and 3 Digits: $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17576000$

$$67600 + 676000 + 1757600 + 17576000 = 20,077,200$$

5. A *palindrome* is a string whose reversal is identical to the original string; one example is the word "radar". How many bit strings of length n are palindromes?

   If n is even, $2^{n/2}$
   If n is odd, $2^{(n+1)/2}$

2 Pigeonhole Principle

1. Show that if five integers are selected from among the eight smallest positive integers, there must be a pair of integers among those selected with a sum equal to 9.

   Group the first eight positive integers into four subsets of two integers each so that the integers of each subset add up to 9. The subsets are $\{1,8\},\{2,7\},\{3,6\}$, and $\{4,5\}$. If five integers are selected from the first eight positive integers, by the pigeonhole principle at least two of them come from the same subset. So, two integers have a sum of 9.

2. How many numbers must be selected from the set $\{1,3,5,7,9,11,13,15\}$ to guarantee that at least one pair of the selected numbers adds up to 16?

   We group these integers into subsets that equal 16. The subsets are $\{1,15\}$, $\{3,13\}$, $\{5,11\}$, and $\{7,9\}$. If five integers are selected from the the set if integers, by the pigeonhole principle at least two of them come from the same subset. So, two integers have a sum of 16.

3. A bowl contains 10 red balls and 10 blue balls. We select and remove a number of balls at random, without looking at them. How many balls must we select to ensure we will have at least three balls of the same color?

   The worst case scenario is picking up alternating colors, so we can only be sure we hold 3 balls of the same color after 5 picks. After picks 1 and 2, we obviously don't have 3 of the same color. After pick 3 we may have 2 of one color and 1 of the other. After pick 4, we may have 2 of one color and 2 of the other. After pick 5, we may have of one color, 4 of one color and 1 of another, or 3 of one color and 2 of another. So, we need to pick 5 to ensure we have 3 of the same color.

4. Let n be a positive integer. Show that in any set of n consecutive integers, there is exactly one divisible by n.

   Let $a, a+1, ..., a+n-1$ be the integers in the sequence. The integers $(a+i) \pmod{n}$, $i = 0,1,2,...,n-1$, are distinct, because $0 < (a+j)-(a+k) < n$ whenever $0 \leq k < j \leq n-1$. There are n possible values for $(a+i) \pmod{n}$ and there are n different integers in the set because each of these values is used exactly once. So, there is exactly one integer in the sequence that is divisible by n.

5. There are 38 different possible time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

$$\left\lceil \frac{677}{38} \right\rceil = 18$$

3 Permutations and Combinations

1. List all the permutations of $\{\text{apple}, \text{banana}, \text{cherry}\}$.

   {apple, banana, cherry}
   {apple, cherry, banana}
   {banana, apple, cherry}
   {banana, cherry, apple}
   {cherry, apple, banana}
   {cherry, banana, apple}

2. How many bit strings contain exactly eight 0s and ten 1s if every 0 must be immediately followed by a 1?

   $C(10, 2) = 45$

3. How many ways are there for a horse race with three distinguishable horses to finish if ties are possible? Note: Two or three horses may tie.

   $P(3, 3) + P(3, 2) + 1 = 6 + 6 + 1 = 13$

4. A circular r-permutation of n people is a seating of r of these n people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table. Find the number of circular 3-permutations of 5 people.

   $P(5, 3) = 60$
   Using the Division Rule $\frac{60}{3} = 20$

5. Seven distinguishable pigs and nine distinguishable sheep are on a farm. How many ways are there to select 5 animals from the farm to be put into a separate pen if at least one pig must be moved to the pen?

   We have $9 + 7 = 16$ animals.
   Pen with any number of Sheep/Pigs:
   $C(16, 5) = 4368$
   Pen with only sheep:
   $C(9, 5) = 126$ Using the subtraction rule for a pen with at least one pig:
   $C(16, 5) - C(9, 5) = 4368 - 126 = 4242$

4 Binomial Coefficients and Identifies

1. Find the expansion of $(x + y)^6$.

   From the Binomial Theorem, we have

$$(x+y)^6 = \sum_{j=0}^{6} \binom{6}{j} x^{6-j} y^j$$

$$= \binom{6}{0} x^6 y^0 + \binom{6}{1} x^5 y^1 + \binom{6}{2} x^4 y^2 + \binom{6}{3} x^3 y^3 + \binom{6}{4} x^2 y^4 + \binom{6}{5} x^1 y^5 + \binom{6}{6} x^0 y^6$$

$$= x^6 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x^1y^5 + y^6$$

2. What is the coefficient of x^7 in $(1 + x)^{11}$?

   From the Binomial Theorem, we have

$$(1+x)^{11} = \sum_{j=0}^{11} \binom{11}{j} 1^{11-j} x^j$$

   We then plug in $j = 7$

$$\binom{11}{7} 1^{11-7} x^7 = 330x^7$$

   So the coefficient is 330.

3. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

   From the Binomial Theorem, we have

$$(2x-3y)^{200} = \sum_{j=0}^{200} \binom{200}{j} (2x)^{200-j}(-3y)^j$$

   We then plug in $j = 99$

$$\binom{200}{99}(2x)^{200-99}(-3y)^{99}$$

$$= \binom{200}{99}(2x)^{101}(-3y)^{99}$$

$$= \binom{200}{99}(2^{101})(-3^{99})x^{101}y^{99}$$

   So the coefficient is $\binom{200}{99}(2^{101})(-3^{99})$.

4. What is the row of Pascal's triangle containing the binomial coefficients $\binom{9}{k}$, $0 \le k \le 9$?

   $\binom{9}{0}$ $\binom{9}{1}$ $\binom{9}{2}$ $\binom{9}{3}$ $\binom{9}{4}$ $\binom{9}{5}$ $\binom{9}{6}$ $\binom{9}{7}$ $\binom{9}{8}$ $\binom{9}{9}$

   1 9 36 84 126 126 84 36 9 1

5. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \le k \le 10$, is:

   1 10 45 120 210 252 210 120 45 10 1

   Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

   1 11 55 165 330 462 462 330 165 55 11 1

5 Generalized Permutations and Combinations

1. How many strings of six uppercase letters are there?

   26^6

2. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$, where x_1, x_2, x_3, and x_4 are nonnegative integers?

   $C(n+r-1,r) = C(4+17-1,17) = C(20,17) = 1140$

3. How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

   $p+n+d+q+h = 20$
   $C(n+r-1,r) = C(5+20-1,20) = C(24,20) = 10626$

4. A bagel shop has onion bagels, poppy seed bagels, egg bagels, everything bagels, blueberry bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose a dozen bagels with at least one of each kind?

   $C(n+r-1,r) = C(8+4-1,4) = C(11,4) = 330$

5. How many positive integers less than 1,000,000 have the sum of their digits equal to 19?

   $a_6 + a_5 + a_4 + a_3 + a_2 + a_1 = 19$
   $C(n+r-1,r) = C(6+19-1,19) = C(24,19) = 42504$
   This answer allows us to have $a > 9$. So we over counted and must use the subtraction rule to get the correct answer.
   There are $C(n+r-1,r) = C(9+6-1,5) = C(14,5) = 2002$ ways that 1 of these digits are greater than 1. We only need to look at one variable being greater than 9 as $10+10 = 20$. So we will never be in that case. However, we have 6 possible options for those that are greater than 1.
   $6 \cdot 2002 = 12012$ $42504 - 12012 = 30492$

6 An Introduction to Discrete Probability

1. What is the probability that a card selected at random from a standard deck of 52 cards is an ace?

$$\frac{4}{52} = 1/13$$

2. What is the probability that a five-card poker hand contains at least one Queen?

$$1 - \frac{C(48,5)}{C(52,5)}$$

3. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

$$\frac{100}{5} = 20$$
$$\left\lfloor \frac{100}{7} \right\rfloor = 14$$
$$\left\lfloor \frac{100}{35} \right\rfloor = 2$$
$$20 + 14 - 2 = 32$$
$$\frac{32}{100} = \frac{8}{25}$$

4. What is the probability that a fair six-sided die never comes up an even number when it is rolled six times?

$$\frac{3}{6}\frac{3}{6}\frac{3}{6}\frac{3}{6}\frac{3}{6}\frac{3}{6} = \frac{1}{64}$$

5. Which is more likely: rolling a total of 9 when two six-sided dice are rolled, or rolling a total of 9 when three six-sided dice are rolled?

   The number of ways to get 9 from 2 dice is 4:

$$\{3,6\}, \{4,5\}, \{5,4\}, \{6,3\}$$

So the probability is $\frac{4}{36} = \frac{1}{9} \approx 0.11111$ The number of ways to get 9 from 3 dice is :

$$\{1,2,6\},\{1,3,5\},\{1,4,4\},\{1,5,3\},\{1,6,2\},$$
$$\{2,1,6\},\{2,2,5\},\{2,3,4\},\{2,4,3\},\{2,5,2\},\{2,6,1\},$$
$$\{3,1,5\},\{3,2,4\},\{3,3,3\},\{3,4,2\},\{3,5,1\},$$
$$\{4,1,4\},\{4,2,3\},\{4,3,2\},\{4,4,1\}$$
$$\{5,1,3\},\{5,2,2\},\{5,3,1\}$$
$$\{6,1,2\},\{6,2,1\}$$

So the probability is $\frac{25}{216} \approx 0.1157$
Therefore, it is more likely to roll a a total of 9 when three six-sided dice are rolled.

7 Probability Theory

1. What probability should be assigned to the outcome of heads when a biased coin is tossed, if heads is three times as likely to come up as tails? What probability should be assigned to the outcome of tails?

   Since heads is three times as likely to come up as tails, $H = 3T$. So,

$$1 = T + 3T$$
$$1 = 4T$$
$$T = \frac{1}{4}$$
$$H = \frac{3}{4}$$

2. What is the probability of these events when we randomly select a permutation of {a, b, c, d}?
   Note that our set of permutations is : { {a, b, c, d} , {a, b, d, c} , {a, c, b, d} , {a, c, d, b} , {a, d, b, c} , {a, d, c, b} , {b, a, c, d} , {b, a, d, c} , {b, c, a, d} , {b, c, d, a} , {b, d, a, c} , {b, d, c, a} , {c, a, b, d} , {c, a, d, b} , {c, b, a, d} , {c, b, d, a} , {c, d, a, b} , {c, d, b, a} , {d, a, b, c} , {d, a, c, b} , {d, b, a, c} , {d, b, c, a} , {d, c, a, b} , {d, c, b, a}}

(a) a precedes d

The probability is $\frac{1}{2}$.

(b) d precedes a

The probability is $\frac{1}{2}$

(c) d precedes a and d precedes b

The probability is $\frac{1}{3}$

(d) d precedes a, d precedes b, and d precedes c

The probability is $\frac{1}{4}$

(e) d precedes c and b precedes a

The probability is $\frac{1}{4}$

3. Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.

Consider, $p(E \cup F) \geq p(E) = 0.7$ and

$$p(E \cup F) \leq 1.$$

Note that

$$p(E) + p(F) - p(E \cap F) \leq 1.$$

So,

$$0.7 + 0.5 - p(E \cap F) \leq 1.$$

Solving for $p(E \cap F)$ gives $p(E \cap F) \geq 0.2$

4. If E and F are independent events, prove or disprove that $\overline{E}$ and F are necessarily independent events.

Consider $E \cup \overline{E}$. This is the entire sample space S, the event F can be split into two disjoint events,

$$F = S \cap F = (E \cup \overline{E}) \cap F = (E \cap F) \cup (\overline{E} \cap F)$$

using the distributive law. Therefore,

$$p(F) = p((E \cap F) \cup (\overline{E} \cap F)) = p(E \cap F) + p(\overline{E} \cap F)$$

because these two events are disjoint. Subtracting $p(E \cap F)$ from both sides, using the fact that $p(E \cap F) = p(E)p(F)$ (from E and F being independent), and factoring, we have

$$p(F)[1 - p(E)] = p(\overline{E} \cap F)$$

Because $1 - p(E) = p(\overline{E})$, this says that

$$p(\overline{E} \cap F) = p(\overline{E})p(F)$$

5. Find the smallest number of people you need to choose at random so that the probability that at least two of them were both born on October 31 exceeds $\frac{1}{2}$.

   The probability that someone is born on October 31 is $\frac{1}{365}$
   The probability that someone is not born on October 31 is $1-\frac{1}{365} = \frac{364}{365}$.
   The probability that no one out of n people is born on October 31 is $(\frac{364}{365})^n$
   The probability that exactly one out of n people is born on October 31 is $n(\frac{1}{365})(\frac{364}{365})^{n-1}$
   The probability that at least two out of n people are born on October 31 is $1 - (\frac{364}{365})^n - n(\frac{1}{365})(\frac{364}{365})^{n-1}$.
   So, we get 613.