Horse Numbers

Definition of Horse Numbers

Recall If n horses run in a race then the number of ways they can finish is $n!$.

Definition of Horse Numbers

Recall If n horses run in a race then the number of ways they can finish is $n!$.

Not Quite

Definition of Horse Numbers

Recall If n horses run in a race then the number of ways they can finish is $n!$.

Not Quite Horses can **tie**.

Definition of Horse Numbers

Recall If n horses run in a race then the number of ways they can finish is $n!$.

Not Quite Horses can **tie**.

Due $H(n)$ is the number of ways n horse can finish a race.

Definition of Horse Numbers

Recall If n horses run in a race then the number of ways they can finish is $n!$.

Not Quite Horses can **tie**.

Due $H(n)$ is the number of ways n horse can finish a race.
$H(0) = 1$ by convention.

Definition of Horse Numbers

Recall If n horses run in a race then the number of ways they can finish is $n!$.

Not Quite Horses can **tie**.

Due $H(n)$ is the number of ways n horse can finish a race.
$H(0) = 1$ by convention. $H(1) = 1$ clearly.

Definition of Horse Numbers

Recall If n horses run in a race then the number of ways they can finish is $n!$.

Not Quite Horses can **tie**.

Due $H(n)$ is the number of ways n horse can finish a race. $H(0) = 1$ by convention. $H(1) = 1$ clearly.

Examples Horses are x_1, x_2.

1. 2 horses: $x_1 < x_2$ OR $x_2 < x_1$ OR $x_1 = x_2$. $H(2) = 3$.

Definition of Horse Numbers

Recall If n horses run in a race then the number of ways they can finish is $n!$.

Not Quite Horses can **tie**.

Due $H(n)$ is the number of ways n horse can finish a race.
$H(0) = 1$ by convention. $H(1) = 1$ clearly.

Examples Horses are x_1, x_2.

1. 2 horses: $x_1 < x_2$ OR $x_2 < x_1$ OR $x_1 = x_2$. $H(2) = 3$.
2. Three horses. Work on it!

Definition of Horse Numbers

Recall If n horses run in a race then the number of ways they can finish is $n!$.

Not Quite Horses can **tie**.

Due $H(n)$ is the number of ways n horse can finish a race. $H(0) = 1$ by convention. $H(1) = 1$ clearly.

Examples Horses are x_1, x_2.

1. 2 horses: $x_1 < x_2$ OR $x_2 < x_1$ OR $x_1 = x_2$. $H(2) = 3$.
2. Three horses. Work on it!

Answer on next slide.

Three Horses

Three Horses

$$x_1 < x_2 < x_3 \qquad x_1 < x_3 < x_2 \qquad x_1 < x_2 = x_3$$

Three Horses

$$x_1 < x_2 < x_3 \qquad x_1 < x_3 < x_2 \qquad x_1 < x_2 = x_3$$

$$x_2 < x_1 < x_3 \qquad x_2 < x_3 < x_1 \qquad x_2 < x_3 = x_1$$

Three Horses

$$
\begin{array}{lll}
x_1 < x_2 < x_3 & x_1 < x_3 < x_2 & x_1 < x_2 = x_3 \\
x_2 < x_1 < x_3 & x_2 < x_3 < x_1 & x_2 < x_3 = x_1 \\
x_3 < x_1 < x_2 & x_3 < x_2 < x_1 & x_3 < x_2 = x_1
\end{array}
$$

Three Horses

$$
\begin{array}{lll}
x_1 < x_2 < x_3 & x_1 < x_3 < x_2 & x_1 < x_2 = x_3 \\
x_2 < x_1 < x_3 & x_2 < x_3 < x_1 & x_2 < x_3 = x_1 \\
x_3 < x_1 < x_2 & x_3 < x_2 < x_1 & x_3 < x_2 = x_1 \\
x_1 = x_2 < x_3 & x_1 = x_3 < x_2 & x_2 = x_3 < x_1
\end{array}
$$

Three Horses

$x_1 < x_2 < x_3$	$x_1 < x_3 < x_2$	$x_1 < x_2 = x_3$
$x_2 < x_1 < x_3$	$x_2 < x_3 < x_1$	$x_2 < x_3 = x_1$
$x_3 < x_1 < x_2$	$x_3 < x_2 < x_1$	$x_3 < x_2 = x_1$
$x_1 = x_2 < x_3$	$x_1 = x_3 < x_2$	$x_2 = x_3 < x_1$
$x_1 = x_2 = x_3$		

Three Horses

$x_1 < x_2 < x_3$ $\quad x_1 < x_3 < x_2$ $\quad x_1 < x_2 = x_3$

$x_2 < x_1 < x_3$ $\quad x_2 < x_3 < x_1$ $\quad x_2 < x_3 = x_1$

$x_3 < x_1 < x_2$ $\quad x_3 < x_2 < x_1$ $\quad x_3 < x_2 = x_1$

$x_1 = x_2 < x_3$ $\quad x_1 = x_3 < x_2$ $\quad x_2 = x_3 < x_1$

$x_1 = x_2 = x_3$

$H(3) = 13$

Four Horses

Work on it

Four Horses

Work on it
Answer on next slide.

Four Horses: Answer in a Way That Can Generalize

Four Horses: Answer in a Way That Can Generalize

1. Pick one of x_1, x_2, x_3, x_4 to be **unique min**: $\binom{4}{1}$. Order the 3 horses left: $H(3)$. Total: $\binom{4}{1}H(3)$

Four Horses: Answer in a Way That Can Generalize

1. Pick one of x_1, x_2, x_3, x_4 to be **unique min**: $\binom{4}{1}$. Order the 3 horses left: $H(3)$. Total: $\binom{4}{1} H(3)$
2. Pick two of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{2}$. Order the 2 horses left: $H(2)$. Total: $\binom{4}{2} H(2)$

Four Horses: Answer in a Way That Can Generalize

1. Pick one of x_1, x_2, x_3, x_4 to be **unique min**: $\binom{4}{1}$. Order the 3 horses left: $H(3)$. Total: $\binom{4}{1}H(3)$
2. Pick two of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{2}$. Order the 2 horses left: $H(2)$. Total: $\binom{4}{2}H(2)$
3. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{3}$. Order the 1 horses left: $H(1)$. Total: $\binom{4}{3}H(1)$

Four Horses: Answer in a Way That Can Generalize

1. Pick one of x_1, x_2, x_3, x_4 to be **unique min**: $\binom{4}{1}$. Order the 3 horses left: $H(3)$. Total: $\binom{4}{1}H(3)$
2. Pick two of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{2}$. Order the 2 horses left: $H(2)$. Total: $\binom{4}{2}H(2)$
3. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{3}$. Order the 1 horses left: $H(1)$. Total: $\binom{4}{3}H(1)$
4. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{4}$. Order the 0 horses left: $H(0)$. Total: $\binom{4}{4}H(0)$

Four Horses: Answer in a Way That Can Generalize

1. Pick one of x_1, x_2, x_3, x_4 to be **unique min**: $\binom{4}{1}$. Order the 3 horses left: $H(3)$. Total: $\binom{4}{1}H(3)$
2. Pick two of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{2}$. Order the 2 horses left: $H(2)$. Total: $\binom{4}{2}H(2)$
3. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{3}$. Order the 1 horses left: $H(1)$. Total: $\binom{4}{3}H(1)$
4. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{4}$. Order the 0 horses left: $H(0)$. Total: $\binom{4}{4}H(0)$

Total:

Four Horses: Answer in a Way That Can Generalize

1. Pick one of x_1, x_2, x_3, x_4 to be **unique min**: $\binom{4}{1}$. Order the 3 horses left: $H(3)$. Total: $\binom{4}{1}H(3)$
2. Pick two of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{2}$. Order the 2 horses left: $H(2)$. Total: $\binom{4}{2}H(2)$
3. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{3}$. Order the 1 horses left: $H(1)$. Total: $\binom{4}{3}H(1)$
4. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{4}$. Order the 0 horses left: $H(0)$. Total: $\binom{4}{4}H(0)$

Total:

$$\binom{4}{1}H(3) + \binom{4}{2}H(2) + \binom{4}{3}H(1) + \binom{4}{4}H(0) = 75$$

Four Horses: Answer in a Way That Can Generalize

1. Pick one of x_1, x_2, x_3, x_4 to be **unique min**: $\binom{4}{1}$. Order the 3 horses left: $H(3)$. Total: $\binom{4}{1}H(3)$
2. Pick two of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{2}$. Order the 2 horses left: $H(2)$. Total: $\binom{4}{2}H(2)$
3. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{3}$. Order the 1 horses left: $H(1)$. Total: $\binom{4}{3}H(1)$
4. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{4}$. Order the 0 horses left: $H(0)$. Total: $\binom{4}{4}H(0)$

Total:

$$\binom{4}{1}H(3) + \binom{4}{2}H(2) + \binom{4}{3}H(1) + \binom{4}{4}H(0) = 75$$

Can write in a nicer way for summations:

Four Horses: Answer in a Way That Can Generalize

1. Pick one of x_1, x_2, x_3, x_4 to be **unique min**: $\binom{4}{1}$. Order the 3 horses left: $H(3)$. Total: $\binom{4}{1}H(3)$
2. Pick two of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{2}$. Order the 2 horses left: $H(2)$. Total: $\binom{4}{2}H(2)$
3. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{3}$. Order the 1 horses left: $H(1)$. Total: $\binom{4}{3}H(1)$
4. Pick three of x_1, x_2, x_3, x_4 to be **only mins**: $\binom{4}{4}$. Order the 0 horses left: $H(0)$. Total: $\binom{4}{4}H(0)$

Total:

$$\binom{4}{1}H(3) + \binom{4}{2}H(2) + \binom{4}{3}H(1) + \binom{4}{4}H(0) = 75$$

Can write in a nicer way for summations:

$$\binom{4}{0}H(0) + \binom{4}{1}H(1) + \binom{4}{2}H(2) + \binom{4}{3}H(3) = 75$$

n Horses

$H(n)$:

n Horses

$H(n)$:

For $1 \le i \le n$, choose i horses to be the **only min**: $\binom{n}{i} = \binom{n}{n-i}$.

n Horses

$H(n)$:

For $1 \leq i \leq n$, choose i horses to be the **only min**: $\binom{n}{i} = \binom{n}{n-i}$.

Order the remaining $n - i$ horses: $H(n - i)$.

n Horses

$H(n)$:

For $1 \leq i \leq n$, choose i horses to be the **only min**: $\binom{n}{i} = \binom{n}{n-i}$.

Order the remaining $n - i$ horses: $H(n - i)$.

$$H(n) = \sum_{i=1}^{n} \binom{n}{n-i} H(n-i) = \sum_{i=0}^{n-1} \binom{n}{i} H(i).$$