BILL AND EMILY RECORD LECTURE!!!!

Increasing and Decreasing Sequences

If you have a Sequence of Length m . . .

Def If $a_1, a_2, \ldots, a_m$ is a sequence of distinct reals then a **subsequence** is a subset of the sequence in the order given.

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Example $1, 3, 10, 8, 20, 5, 2$
Increasing Subsequence: 1,3,10,20.
Decreasing Subsequence: 10,8,5,2.

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Example $1, 3, 10, 8, 20, 5, 2$
Increasing Subsequence: 1,3,10,20.
Decreasing Subsequence: 10,8,5,2.

A sequence has the $ID(k)$ (Increasing-Decreasing) property if there is **either** an increasing subsequence of length k OR a decreasing subsequence of length k.

Work on In Groups

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- Try to find pattern!

Answers: $k = 2$ Case is Easy

The sequence

$$1$$

does not satisfy $ID(2)$.

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The sequence

$$1$$

does not satisfy $ID(2)$.
ANY sequence of length 2 DOES satisfy $ID(2)$.
$X_2 = 2$.

Answers: $k = 3$ Case

The sequence

$$3, 6, 1, 4$$

does not satisfy $ID(3)$.

Answers: $k = 3$ Case

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T or F: Every seq of length 5 have $ID(3)$.

Answers: $k = 3$ Case

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T or F: Every seq of length 5 have $ID(3)$.
Does every sequence of length 5 satisfy ID(3)?

Answers: $k = 3$ Case

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T or F: Every seq of length 5 have $ID(3)$.
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Yes. Can prove by messy cases OR see next slide for proof by Pigeonhole Principle.

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- An INC subseq of length 1. Let $u_1 = 1$.
- A DEC subseq of length 1. Let $d_1 = 1$.

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IF subseq stopped at the FOURTH element there would be

- An INC subseq of length 2. Let $u_4 = 2$.
- A DEC subseq of length 2. Let $d_4 = 2$.

Answers: $k = 3$ Case by P. Principle

Look at $3, 5, 1, 4, 2$.

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- A DEC subseq of length 2. Let $d_4 = 2$.

IF subseq stopped at the FIFTH element there would be

- An INC subseq of length 2. Let $u_5 = 2$.
- A DEC subseq of length 2. Let $d_5 = 3$.

Answers: $k = 3$ Case by P. Principle

- Let u_i be length of longest INC subseq that ends at a_i.
- Let d_i be length of longest DEC subseq that ends at a_i.

Lemma If $i < j$ then $(u_i, d_i) \neq (u_j, d_j)$.
Pf
If $a_i < a_j$ then u_i goes up by at least 1.
If $a_i > a_j$ then e_i goes up by at least 1.
End of Proof

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FIVE elements map into FOUR elements, so by P. Princ, some elements is mapped to twice.

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Prove in groups

Thm Let $k \geq 3$. Let $n = XXX(k)$. Any sequence of distinct numbers of length n has $ID(k)$.

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$(k-1)^2 + 1$ elements map into $(k-1)^2$ elements, so by P. Princ, some elements is mapped to twice.
CANNOT happen by Lemma.

Is $(k-1)^2+1$ tight?

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YES. there IS a sequence of length $(k-1)^2$ where NOT $ID(k)$.
EXAMPLE
$k=4$
$4,3,2,1 \qquad 8,7,6,5 \qquad 12,11,10,9 \qquad 15,14,13,12.$

Generalized Pigeonhole Principle

I want Three Balls in the Same Box!

If there are 101 balls going into 100 boxes what do we know?

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some box gets ≥ 2 balls.

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I want Three Balls in the Same Box!

If there are 101 balls going into 100 boxes what do we know?
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If there are 102 balls going into 100 boxes what do we know? still:
some box gets ≥ 2 balls.

I want Three Balls in the Same Box!

If there are 101 balls going into 100 boxes what do we know?
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If there are 102 balls going into 100 boxes what do we know? still:
some box gets ≥ 2 balls.

How many balls do you need before you are guaranteed ≥ 3 balls in a box?

I want Three Balls in the Same Box!

If there are 101 balls going into 100 boxes what do we know?
some box gets ≥ 2 balls.

If there are 102 balls going into 100 boxes what do we know? still:
some box gets ≥ 2 balls.

How many balls do you need before you are guaranteed ≥ 3 balls in a box?

If there are 200 balls going into 100 boxes what do we know?

I want Three Balls in the Same Box!

If there are 101 balls going into 100 boxes what do we know?
some box gets ≥ 2 balls.

If there are 102 balls going into 100 boxes what do we know? still:
some box gets ≥ 2 balls.

How many balls do you need before you are guaranteed ≥ 3 balls in a box?
If there are 200 balls going into 100 boxes what do we know? still:
some box gets ≥ 2 balls.

I want Three Balls in the Same Box!

If there are 101 balls going into 100 boxes what do we know?
some box gets ≥ 2 balls.

If there are 102 balls going into 100 boxes what do we know? still:
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How many balls do you need before you are guaranteed ≥ 3 balls in a box?

If there are 200 balls going into 100 boxes what do we know? still:
some box gets ≥ 2 balls.

If there are 201 balls going into 100 boxes what do we know?

I want Three Balls in the Same Box!

If there are 101 balls going into 100 boxes what do we know?
some box gets ≥ 2 balls.

If there are 102 balls going into 100 boxes what do we know? still:
some box gets ≥ 2 balls.

How many balls do you need before you are guaranteed ≥ 3 balls in a box?

If there are 200 balls going into 100 boxes what do we know? still:
some box gets ≥ 2 balls.

If there are 201 balls going into 100 boxes what do we know?
AH-HA: some box gets ≥ 3 balls.

General Case

If you have m balls in n boxes then some box has at least $XXX(n, m)$ balls.
Find $XXX(n, m)$ in groups.

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Lets try this out:
101 ball in 100 boxes: $\lceil 101 \rceil 100 = 2$.

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101 ball in 100 boxes: $\lceil 101 \rceil 100 = 2$.

200 ball in 100 boxes: $\lceil 200 \rceil 100 = 2$.

201 ball in 100 boxes: $\lceil 201 \rceil 100 = 3$.

Application to Geometry

1. If there are 5 points in the unit square there must be 2 that are XXX apart?
2. If there are 6 points in the unit square there must be 2 that are XXX apart?
3. $\vdots$

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5 points: Divide the square into 4 squares (whiteboard). Two of the points must be in the same small square, which has diagonal $\frac{\sqrt{2}}{2}$.

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5 points: Divide the square into 4 squares (whiteboard). Two of the points must be in the same small square, which has diagonal $\frac{\sqrt{2}}{2}$.

For which n will we get 5 in the same small square?
We are putting n points into 4 boxes and want some box to have 5.
Want $\lceil \frac{m}{4} \rceil = 5$. Take $m = 17$.
Then have 5 points in $\frac{1}{2} \times \frac{1}{2}$
FINISH AT WHITEBOARD