Small Ramsey Numbers

Exposition by William Gasarch

April 30, 2024

FILL OUT YOUR EVALS FOR ALL YOUR COURSES

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3. Used by the Dept to put together teaching reports for faculty for tenure and full prof cases. I have written such reports.

Lets Party Like Its January of 2019

Recall the first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

> *If there are 6 people at a party, either 3 know each other or 3 do not know each other.*

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Recall the first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

If there are 6 people at a party, either 3 know each other or 3 do not know each other.

We state this in terms of colorings of edges of graphs.

For all 2-coloring of the edges of K_6 *there is a mono* K_3*.*

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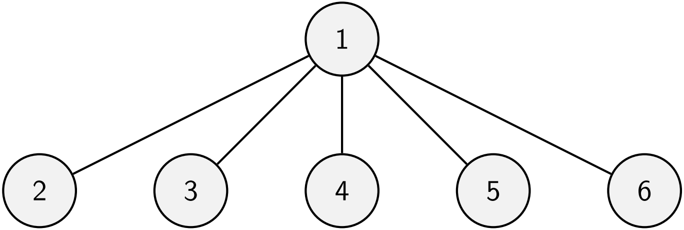

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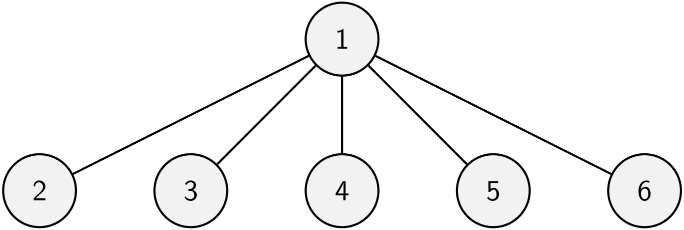

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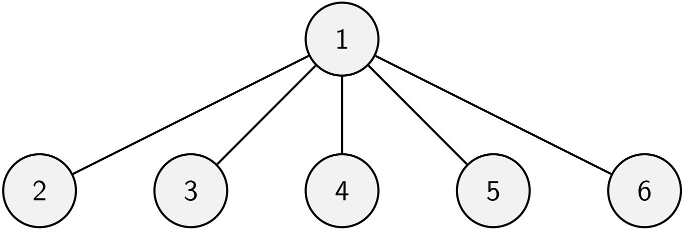

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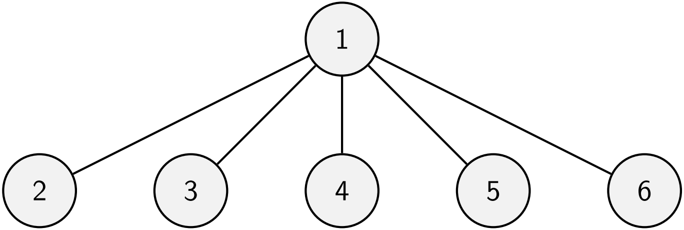

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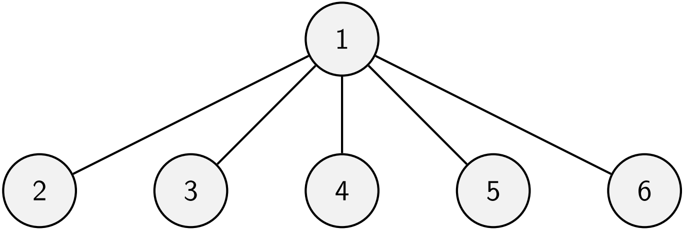
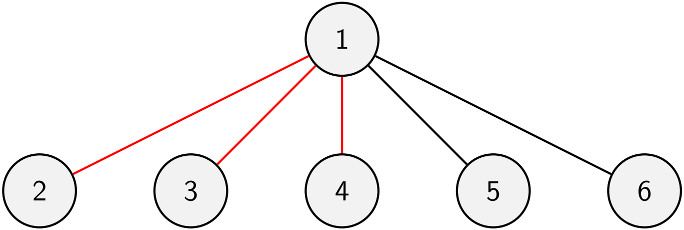
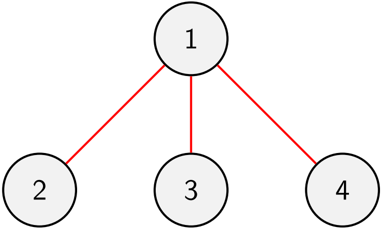


There are 5 edges coming out of vertex 1.
They are 2 colored.
$\exists$ 3 edges from vertex 1 that are the same color.
We can assume $(1, 2)$, $(1, 3)$, $(1, 4)$ are all **RED**.

(1,2), (1,3), (1,4) are RED



We Look Just at Vertices 1,2,3,4



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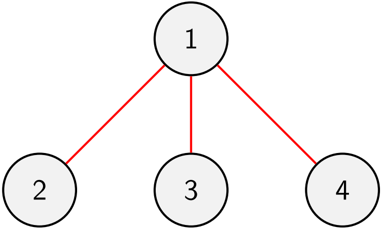

If $(2,3)$ is **RED** then get **RED** Triangle. So assume (2,3) is **BLUE**.

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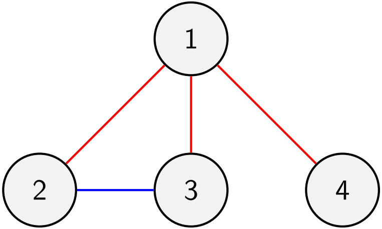

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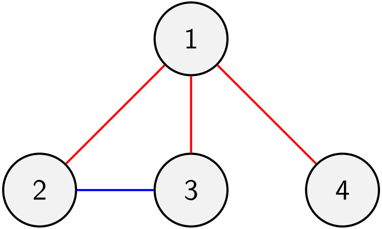

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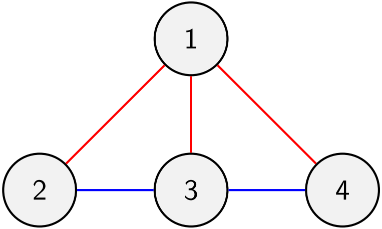

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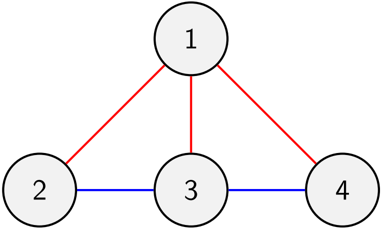

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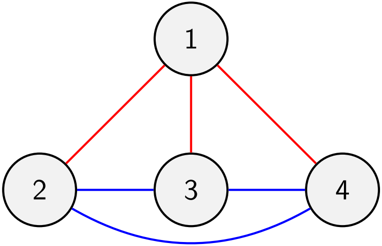

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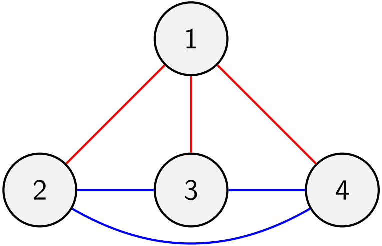

Note that there is a **BLUE** triangle with vertices $2, 3, 4$. Done!

What if we color edges of K_5?

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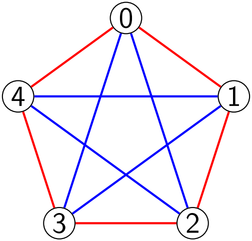

This graph is not arbitrary.
$SQ_5 = \{x^2 \pmod 5 : 0 \le x \le 4\} = \{0, 1, 4\}$.

- If $i - j \in SQ_5$ then **RED**.
- If $i - j \notin SQ_5$ then **BLUE**.

Asymmetric Ramsey Numbers

Definition $R(a, b)$ is least n such that for all 2-colorings of K_n there is **either** a red K_a or a blue K_b.

1. $R(a, b) = R(b, a)$.
2. $R(2, b) = b$
3. $R(a, 2) = a$

$R(a,b) \le R(a-1,b) + R(a,b-1)$

Theorem $R(a,b) \le R(a-1,b) + R(a,b-1)$
Proof
Let $n = R(a-1,b) + R(a,b-1)$. $\mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]$.
Case 1 $(\exists v)[\deg_R(v) \ge R(a-1,b)]$. Look at the $R(a-1,b)$ vertices that are **RED** to v. By Definition of $R(a-1,b)$ either

- There is a **RED** K_{a-1}. Combine with v to get **RED** K_a.
- There is a **BLUE** K_b.

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Case 2 $(\exists v)[\deg_B(v) \geq R(a, b-1)]$. Similar to Case 1.

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Case 3
$(\forall v)[\deg_R(v) \le R(a-1,b) - 1 \wedge \deg_B(v) \le R(a,b-1) - 1]$
$(\forall v)[\deg(v) \le R(a-1,b) + R(a,b-1) - 2 = n-2]$
Not possible since every vertex of K_n has degree $n-1$.

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$
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Theorem $R(3,4) \le 9$.
Let COL be a 2-coloring of the edges of K_9.
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Case 2 $(\exists v)[\deg_B(v) \geq 6]$. $v_1, v_2, v_3, v_4, v_5, v_6$ are **BLUE** to v.
Either:
(1) a **RED** K_3, or
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NOTE Can't have any $\deg_R(v) \leq 2$.

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Case 3 $(\forall v)[\deg_R(v) = 3]$. The **RED** subgraph has 9 nodes each of degree 3. Impossible!

Reminder of the Odd Vertex Things

Lemma Let $G = (V, E)$ be a graph.

$$V_{\mathrm{even}} = \{v : \deg(v) \equiv 0 \pmod 2\}$$
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Then $|V_{\mathrm{odd}}| \equiv 0 \pmod 2$.

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$$\sum_{v \in V_{\mathrm{odd}}} \deg(v) \equiv 0 \pmod 2.$$

Sum of odds $\equiv 0 \pmod 2$. Must have even numb of them. So $|V_{\mathrm{odd}}| \equiv 0 \pmod 2$.

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Key: $R(2,4)$ and $R(3,3)$ were both **even!**

Theorem $R(a,b) \leq$

1. $R(a,b-1) + R(a-1,b)$ always.
2. $R(a,b-1) + R(a-1,b) - 1$ if $R(a,b-1) \equiv R(a-1,b) \equiv 0 \pmod 2$

Some Better Upper Bounds

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6.$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 - 1 = 9.$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 9 = 14.$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 14 - 1 = 19.$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 19 = 26$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 9 + 9 = 18.$
- $R(4,5) \leq R(3,5) + R(4,4) \leq 14 + 18 - 1 = 31.$
- $R(5,5) \leq R(4,5) + R(5,4) = 62.$

Are these tight?

$R(3,3) \geq 6$

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Note $-1 = 2^2 \pmod 5$. Hence $a - b \in SQ$ iff $b - a \in SQ$. So the coloring is well defined.

$R(3,3) \geq 6$

$COL(a,b) =$ **RED** if $a-b \equiv SQ \pmod 5$, **BLUE** OW.

- Squares mod 5: 1,4.
- If there is a **RED** triangle then $a-b$, $b-c$, $c-a$ all SQ's. SUM is 0. So

$$x^2+y^2+z^2 \equiv 0 \pmod 5 \text{ Can show impossible}$$

- If there is a **BLUE** triangle then $a-b$, $b-c$, $c-a$ all non-SQ's. Product of nonsq's is a sq. So $2(a-b)$, $2(b-c)$, $2(c-a)$ all squares. SUM to zero- same proof.

UPSHOT $R(3,3) = 6$ and the coloring used math of interest!

$R(4,4) = 18$

$R(4,4) \geq 18$: Need coloring of K_{17} w/o mono K_4.

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Vertices are $\{0, \ldots, 16\}$.

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Same idea as above for K_5, but more cases.
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$R(3,5) = 14$

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Vertices are $\{0,\ldots,13\}$.

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$COL(a,b) =$ **RED** if $a - b \equiv CUBE \pmod{14}$, **BLUE** OW.

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$R(5,5)$– I will give you a paper to read on that soon.

Revisit those Numbers

Int means Interesting Math. Bor means Boring Math.

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- $R(3,4) \le 9$. TIGHT. Int
- $R(3,5) \le 14$. TIGHT. Int
- $R(3,6) \le 19$. KNOWN: 18. Upper Bd Bor, Lower Bd Int
- $R(3,7) \le 26$. KNOWN: 23. Upper Bd Bor, Lower Bd Int
- $R(4,4) \le 18$. TIGHT. Int
- $R(4,5) \le 31$. KNOWN: 25. Both bd Bor
- $R(5,5) \le 62$. KNOWN: Will see it in the paper I give out.

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2. Seemed like a nice **Math** problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.