Induction Examples

250H

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Use induction to prove that $b_n = 3^n - 1$ for all $n \geq 1$.

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By our IH,

$$3b_k + 2 = 3(3^k - 1) + 2$$

$$= 3^{k+1} - 3 + 2$$

$$= 3^{k+1} - 1.$$

Therefore, by PMI our statement holds. ☽

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Inductive Hypothesis: Assume for some odd integer $k \geq 1$, $k^2 - 1 \equiv 0 \bmod 8$.

Inductive Step: Let $a = k+2$. Then,

$$a^2 - 1 = (k+2)^2 - 1$$

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By our IH,

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Note we know k is odd. Therefore, k = 2h+1 for $h \in \mathbf{Z}$.

$$\equiv 4(2h+1) + 4 \mod 8$$

$$\equiv 8h + 8 \mod 8$$

$$\equiv 0 \mod 8.$$

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  - Instead of looking at the k+1 term, we look at the k+2 term

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Inductive Step: Let n = k+2. Then,

$$n^3 + 2n = (k+2)^3 + 2(k+2)$$

$$=k^3 + 6k^2 + 12k + 8 + 2k +4$$

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$$\equiv(0) + 6k^2 + 12k + 12$$

$$\equiv 0 \bmod 3$$

Therefore by PMI, our statement holds. ☽

Prove that $n^3 + 2n$ is divisible by 3 for negative even integers n.

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Inductive Hypothesis: Assume for some even integer $k \leq -2$, $n^3 + 2n \equiv 0 \bmod 3$.

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Inductive Hypothesis: Assume for some even integer $k \leq -2$, $n^3 + 2n \equiv 0 \mod 3$.

Inductive Step: Let $n = k-2$. Then,

$$n^3 + 2n = (k-2)^3 + 2(k-2)$$

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$$=(k^3 + 2k) - 6k^2 + 12k - 12$$

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By our IH,

$$\equiv (0) - 6k^2 + 12k - 12$$

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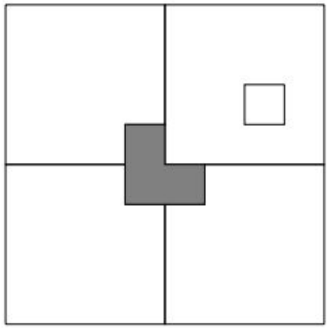
Prove that $n^3 + 2n$ is divisible by 3 for $n = 0$.

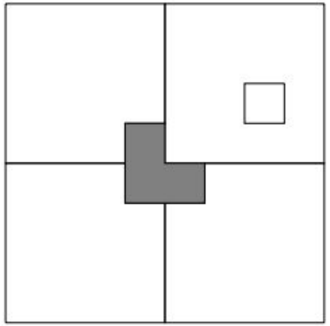
Let $n = 0$. Then $n^3 + 2n \equiv (0)^3 + 2(0) \equiv 0 \mod 3$. ☽

Prove that a 2^n x 2^n chess board with any one square removed can always be covered by L shaped tiles



Prove that a 2^n x 2^n chess board with any one square removed can always be covered by L shaped tiles

Base Case: Let n = 0. So, we have a single square chessboard. If we remove one square then the board is empty. Hence, it is also covered and our base case holds.



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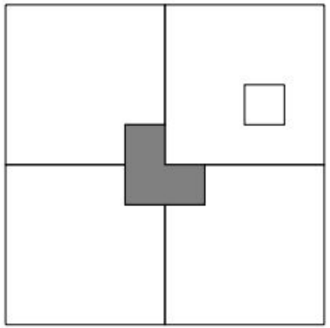

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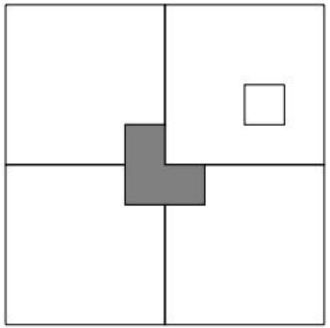
Inductive Step: Consider a 2^n x 2^n chessboard with a missing square. Divide the board into four quarters. Place a L tile in the center so that each quarter is missing a square.



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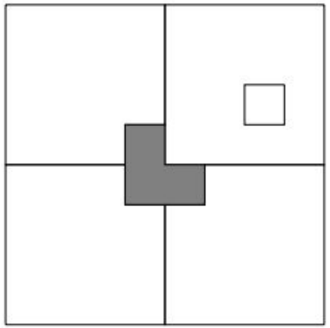
By our inductive hypothesis, each of the quarters can be tiled, which gives us a way to tile a 2^n x 2^n chessboard. Hence, by PMI, we can tile a 2^n x 2^n chessboard. ☽



A chocolate bar consists of unit squares arranged in an n x m rectangular grid. You may split the bar into individual unit squares, by breaking along the lines. Show the number of breaks needed is nm-1.

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Inductive Step: Without loss of generality, break the bar along a row. So, we get a n_1 x m and a n_2 x m bar, where $n_1 + n_1 = n$. By the induction hypothesis, the number of breaks that we need for the two new bars is $n_1m - 1$ and $n_2m - 1$ respectfully.

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$$1 + n_1m - 1 + n_2m - 1$$

$$= n_1m + n_2m - 1$$

$$= m(n_1 + n_2) - 1$$

$$= mn - 1$$

Therefore by PMI, our statement holds. ☽