Arithmetic-Mean Geometric-Mean Inequalities

AM and GM

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How do AM and GM compare when $x_1, \ldots, x_n \in \mathbb{R}^+$?

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The AM-GM Theorem

Thm For all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n \in \mathbb{R}^+$

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Equality happens iff $x_1 = \cdots = x_n$.

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From these implications we easily obtain $(\forall n)[P(n)]$.

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Next Slide

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Note This is AM of 2 numbers! We use AM-GM-2 on it!

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$$\geq (\prod_{i=1}^{2^n} x_i)^{1/2^{n-1}})^{1/2} = (\prod_{i=1}^{2^n} x_i)^{1/2^n}.$$

$n < m$: $P(m) \implies P(n)$

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you can reach any $n \in \mathbb{N}$, then $(\forall n)[P(n)]$.