

START

RECORDING

Constructive Induction

CMSC 250

Introductory Example

- We already know that

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

- **But how? Who told us this?**
- This is **not** how math works; we usually do **not** know the answer ahead of time!

Making a Good Guess with Calculus

- Calculus tells us that (discrete) sums are approximations of (continuous) integrals.
- Then, we can observe that:

$$\sum_{i=1}^n i \approx \int_1^n x \, dx = \frac{1}{2}n^2 + c, \quad c \in \mathbb{R}$$

- So we know that the sum ought to be *some quadratic function of n* .

Making a Good Guess with CS

- Another way to guess the quadratic form would be with **plotting!**
- Suppose $f(n) = \sum_{i=1}^n i$. Then:
 - $f(0) = \sum_{i=1}^0 i = 0$
 - $f(1) = \sum_{i=1}^1 i = 1$
 - $f(2) = \sum_{i=1}^2 i = 1 + 2 = 3$
 - $f(3) = \sum_{i=1}^3 i = 1 + 2 + 3 = 6$
 - ...
 - $f(30) = \sum_{i=1}^{30} i = 1 + 2 + \dots + 30 = 465$
- We can then **fit a curve** and see the quadratic curve by ourselves!

Making a Good Guess

- We saw that the sum is some quadratic polynomial. This is all we know!
- So $\sum_{i=1}^n i$ is some *poly*(n) with degree 2, i.e

$$\sum_{i=1}^n i = An^2 + Bn + C, \quad A, B, C \in \mathbb{R}$$

- *How to determine A, B, and C?*

General Logic

- Solve **as if** you had an inductive proof (so IB, IH, IS)
- For every step, we will establish **conditions** on A, B,C **such that** the relevant step is correct.
 - Contrast this with **directly proving** that every step is correct.

Constant C

- IB: LHS is $\sum_{i=1}^0 i = 0$. For RHS to be equal to LHS we need:

$$An^2 + Bn + C = 0 \Rightarrow C = 0$$

- So we already know that $C = 0$.

Co-efficients A, B

- IH: Assume that the proposition holds for $n \geq 0$. Then:

$$\sum_{i=1}^n i = An^2 + Bn$$

- IS: We want to prove that

$$\left(\sum_{i=1}^n i = An^2 + Bn \right) \Rightarrow \left(\sum_{i=1}^{n+1} i = A(n+1)^2 + B(n+1) \right)$$

Co-efficients A, B

- IH: Assume that the proposition holds for $n \geq 0$. Then:

$$\sum_{i=1}^n i = An^2 + Bn \quad \rightarrow \quad P(n)$$

- IS: We want to prove that

$$\underbrace{\left(\sum_{i=1}^n i = An^2 + Bn \right)}_{P(n)} \Rightarrow \underbrace{\left(\sum_{i=1}^{n+1} i = A(n+1)^2 + B(n+1) \right)}_{P(n+1)}$$

Co-efficients A, B

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^n i + (n+1) \stackrel{\text{IH}}{=} An^2 + Bn + (n+1)$$

- We have to equate this to $A(n+1)^2 + B(n+1)$, since this is what we're trying to prove:

$$\begin{aligned} An^2 + Bn + (n+1) &= A(n+1)^2 + B(n+1) \Rightarrow \\ \cancel{An^2} + \cancel{Bn} + (n+1) &= \cancel{An^2} + 2An + A + \cancel{Bn} + B \Rightarrow \\ n+1 &= 2An + (A+B) \end{aligned}$$

Co-efficients A, B

$$n + 1 = 2An + (A + B)$$

- This is an equality between polynomials of k , so equating the coefficients yields:

$$1 = 2A$$

$$A + B = 1$$

Co-efficients A, B

$$n + 1 = 2An + (A + B)$$

- This is an equality between polynomials in n , so equating the coefficients yields:

$$\begin{aligned}1 &= 2A \\ A + B &= 1\end{aligned}$$

- Note: The IS did not end up with **TRUE**, but with conditions on A, B **for it to be TRUE**.

All Our Constraints

1. $C = 0$

2. $A + B = 1$

3. $2 \cdot A = 1$

• Algebra yields $A = B = 1/2$, so:

$$\sum_{i=0}^n i = \frac{1}{2}n^2 + \frac{1}{2}n + 0 = \frac{n(n+1)}{2}$$

What if Our Guess is Wrong (Over)?

1. Suppose we guess

$$\sum_{i=1}^n i = A \cdot n^3 + B \cdot n^2 + C \cdot n + D$$

2. **This still works**, we will just find $A = 0$ (try it at home!)

What if Our Guess is Wrong (Under)?

1. Suppose we guess

$$\sum_{i=1}^n i = A \cdot n + B$$

2. **This does not work (infeasible equation)**, no $A, B \in \mathbb{R}$ will satisfy the constraints (try it at home!)

Another Example (with Bounds!)

- Let a be a sequence defined as follows:

$$a_n = \begin{cases} 2, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, & n \geq 2 \end{cases}$$

- Task: Find an upper bound for a_n .

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- What kind of inductive structure am I expecting?

Weak

Strong

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Weak

Strong

An inductive base with > 1 elements and a recursive rule with references to two prior terms hints towards strong induction...

Key Step

$$a_n = \begin{cases} 2, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, & n \geq 2 \end{cases}$$

- Because of our experience with sequences like Fibonacci, Tribonacci that all have this form, we **suspect**:

$$a_n \leq C \cdot D^n, \quad C, D \in \mathbb{R}$$

Constraints on C

- IB:

- $a_0 \leq C \cdot D^0 \Leftrightarrow 2 \leq C$

- $a_1 \leq C \cdot D^1 \Leftrightarrow 50 \leq C \cdot D$

Inductive Hypothesis

- IB:
 - $a_0 \leq C \cdot D^0 \Leftrightarrow 2 \leq C$
 - $a_1 \leq C \cdot D^1 \Leftrightarrow 50 \leq C \cdot D$
- IH: Let $n \geq 1$. Assume that $(\forall i \in \{0, 1, 2, \dots, n\})[a_i \leq C \cdot D^i]$

Inductive Step

- IB:

- $a_0 \leq C \cdot D^0 \Leftrightarrow 2 \leq C$

- $a_1 \leq C \cdot D^1 \Leftrightarrow 50 \leq C \cdot D$

- IH: Let $n \geq 1$. Assume that $\forall i \in \{0, 1, 2, \dots, n\}, a_i \leq C \cdot D^i$.

- IS:

$$(\forall i \in \{0, 1, 2, \dots, n\})[a_i \leq C \cdot D^i] \Rightarrow (a_{n+1} \leq C \cdot D^{n+1})$$

Inductive Step

- IS:

$$(\forall i \in \{0, 1, 2, \dots, n\})[a_i \leq C \cdot D^i] \Rightarrow (a_{n+1} \leq C \cdot D^{n+1})$$

- From the definition of a , we have $a_{n+1} = 10a_n + 3a_{n-1}$. Therefore,

$$a_{n+1} = 10a_n + 3a_{n-1} \leq 10 \cdot C \cdot D^n + 3 \cdot C \cdot D^{n-1} \text{ (By IH)}$$

- Want $10 \cdot C \cdot D^n + 3 \cdot C \cdot D^{n-1} \leq C \cdot D^{n+1}$

Inductive Step

- Want

$$\begin{aligned} 10 \cdot \cancel{C} \cdot D^n + 3 \cdot \cancel{C} \cdot D^{n-1} &\leq \cancel{C} \cdot D^{n+1} \Leftrightarrow \\ 10 \cdot D^n + 3 \cdot D^{n-1} &\leq D^{n+1} \end{aligned}$$

- Dividing both sides by D^{n-1} yields:

$$10D + 3 \leq D^2$$

All Constraints

1. $2 \leq C$

2. $50 \leq C \cdot D$

3. $10D + 3 \leq D^2$

- We deal with constraint 3 first.
 - Smallest $D \in \mathbb{R}^{>0}$ that satisfies it:

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- Smallest $D \in \mathbb{R}^{>0}$ that satisfies it: NO, WE ARE BUSY PEOPLE AND WE DON'T WANT TO SPEND TIME SOLVING $D^2 - 10D - 3 \geq 0$

- Smallest $D \in \mathbb{N}$ that satisfies it: $D = \dots ???$ (FIND ONE REAL QUICK, PLZ)

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$D = 11$ works!

All Constraints

1. $2 \leq C$

2. $50 \leq C \cdot D$

3. $10D + 3 \leq D^2$

- Constraint (3) satisfied when $D \geq 11$ (just discussed)

- Since we want to find **tight** bounds for a_n , to minimize C , we select

$D = 11$ and from constraint (2) we have: $50 \leq C \cdot 11 \Leftrightarrow C \geq 4.55 \Rightarrow C_{min} = 4.55$

All Constraints

1. $2 \leq C$

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- Constraint (3) satisfied when $D \geq 11$ (just discussed)

- Since we want to find **tight** bounds for a_n , to minimize C , we select

$D = 11$ and from constraint (2) we have: $50 \leq C \cdot 11 \Leftrightarrow C \geq 4.55 \Rightarrow C_{min} = 4.55$

- Conclusion:

$$a_n \leq 4.55 \cdot 11^n$$

Work on This

- A slight modification on the previous sequence:

$$a_n = \begin{cases} 10, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, & n \geq 2 \end{cases}$$

- Assuming that we still suspect $a_n \leq C \cdot D^n$, **you** solve for the new C, D **right now!**

Work on This

- A slight modification on the previous sequence:

$$a_n = \begin{cases} 10, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, & n \geq 2 \end{cases}$$

- Assuming that we still suspect $a_n \leq C \cdot D^n$, solve for the new C, D !
- Your solution ought to be $C = 10, D = 11$. **What do you observe?**

Coin Problem

- In [Celestia](#), there are only $7c$ and $10c$ coins.
- We want to find the *least monetary amount* payable **exclusively** with such coins!
- In quantifiers (all quantifications assumed over \mathbb{N})

$$(\forall n \geq A)(\exists n_1, n_2)[n = 7n_1 + 10n_2]$$

- **Goal: Find constraints on A via constructive induction!**
- IB: ???

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- **Goal: Find constraints on A via constructive induction!**
- IB: **Defer for later!!!**
- IH: Assume that for $n \geq A$, $(\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]$

Coin Problem (IS)

- From the IH we have $(\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]$
- How can we add/remove coins to get another cent?

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- How can we add/remove coins to get another cent?
 1. $n_2 \geq 2$: Remove two 10c coins, add three 7c coins

$$\begin{aligned}n + 1 &= 7n_1 + 10n_2 + 1 = 7n_1 + 10n_2 + (21 - 20) \\ &= 7(n_1 + 3) + 10(n_2 - 2)\end{aligned}$$

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2. $n_1 \geq 7$: Remove seven 7c coins, add five 10c coins

$$\begin{aligned}n + 1 &= 7n_1 + 10n_2 + 1 = 7n_1 + 10n_2 + (50 - 49) \\ &= 7(n_1 - 7) + 10(n_2 + 5)\end{aligned}$$

Coin Problem (IS)

3. $(n_1 \leq 6) \wedge (n_2 \leq 1)$: Max value is $6 \times 7 + 1 \times 10 = 52$, so $n \leq 52$.

RECAP

- We've shown that if $n \geq 53$, then

$$((\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]) \Rightarrow ((\exists n_1, n_2)[n + 1 = 7 \cdot n_1 + 10n_2])$$

- For which n do we know that $((\exists a, b \in \mathbb{N})[n = 7a + 10b])$?

$\forall n \geq 52$

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Something
Else

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Something
Else

*Only the implication holds! We don't have any **hard truth** (base) about whether it **EVER** holds.*

Coin Problem (IS)

3. $(n_1 \leq 6) \wedge (n_2 \leq 1)$: Max value is $6 \times 7 + 1 \times 10 = 52$, so $n \leq 52$.

- Condition: $A \geq 53$.
- **Now** I need a base case.
- $(\exists? n_1, n_2 \in \mathbb{N})[53 = 7 \cdot n_1 + 10n_2]$

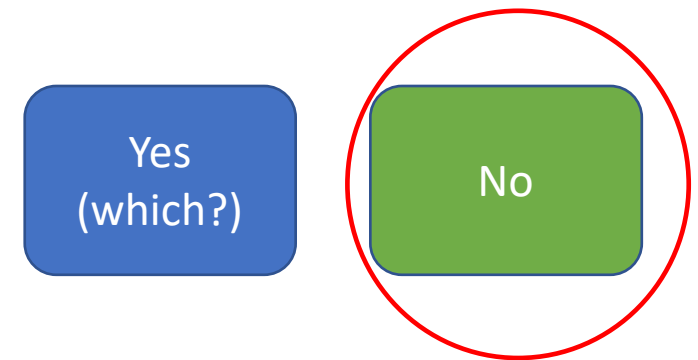
Yes
(which?)

No

Coin Problem (IS)

3. $(n_1 \leq 6) \wedge (n_2 \leq 1)$: Max value is $6 \times 7 + 1 \times 10 = 52$, so $k \leq 52$.

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- $(\exists? n_1, n_2 \in \mathbb{N})[53 = 7 \cdot n_1 + 10n_2]$



Prove it at home (use cases)

Coin Problem (IS)

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- $(\exists? n_1, n_2 \in \mathbb{N})[53 = 7 \cdot n_1 + 10n_2]$
- $(\exists? n_1, n_2 \in \mathbb{N})[54 = 7 \cdot n_1 + 10n_2]$

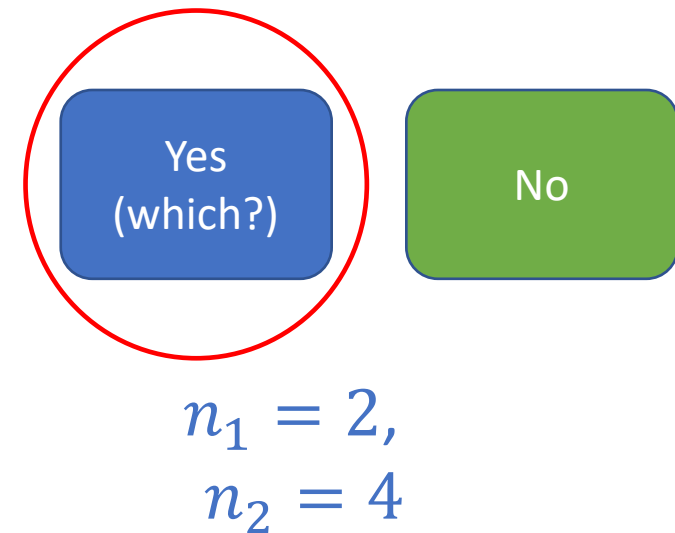
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- We've shown that if $n \geq 53$, then

$$((\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]) \Rightarrow ((\exists n_1, n_2)[n + 1 = 7 \cdot n_1 + 10n_2])$$

- We've also shown that $(\exists r_1, r_2 \in \mathbb{N})[54 = 7r_1 + 10r_2]$
 $(r_1 = 2, r_2 = 4)$

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- We've also shown that $(\exists r_1, r_2 \in \mathbb{N})[54 = 7r_1 + 10r_2]$
 $(r_1 = 2, r_2 = 4)$
- What do we know **NOW** about the theorem?

True for
 $n \geq 52$

True for
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True for
 $n \geq 54$

Nothing

RECAP

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- We've also shown that $(\exists r_1, r_2 \in \mathbb{N})[54 = 7r_1 + 10r_2]$
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- What do we know **NOW** about the theorem?

True for
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What is A ?

- Recall the theorem (all quantifiers over \mathbb{N}):

$$(\forall n \geq A)(\exists n_1, n_2)[n = 7n_1 + 10n_2]$$

- Our goal was to find A .
- $A = 54$ works, and is **optimal, since** $A = 53$ does not work.

Question

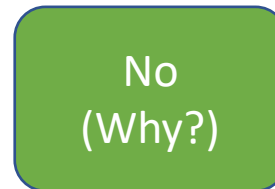
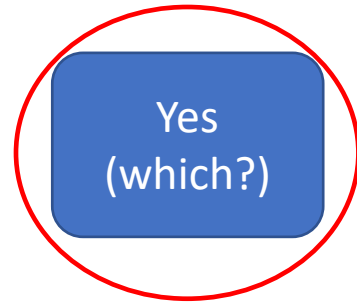
- Is the theorem true for *any* $n \leq 53$?

Yes
(which?)

No
(Why?)

Question

- Is the theorem true for *any* $n \leq 53$?



0, 7, 10, 14, 17, 20, 21, 24, 27, 28, 30, 31, 34, 35, 37, 38, 40,
41, 42, 44, 45, 47, 48, 49, 50, 51, 52

- Note that there are **gaps** between these integers!

And Here's Another

- Let a be a sequence defined as follows:

$$a_n = \begin{cases} 0, & n = 0 \\ 2, & n = 1 \\ a_{\lfloor \frac{n}{2} \rfloor} + a_{\lfloor \frac{n}{4} \rfloor} + 5n, & n \geq 2 \end{cases}$$

- Then, find $C \in \mathbb{R}$ such that

$$(\forall n \in \mathbb{N}) [a_n \leq C \cdot n]$$

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- We proceed via **strong induction** on n .

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- Then, find $C \in \mathbb{R}$ such that

$$(\forall n \in \mathbb{N}) [a_n \leq C \cdot n]$$

- We proceed via **strong induction** on n .
- In fact, to make some of the math easier, we will assume the hypothesis until $P(n-1)$ and prove the step for $P(n)$ instead of $P(n+1)$

Finding C

- IB:
 - For $n = 0$, $a_0 \leq C \cdot 0 \Leftrightarrow 0 \leq 0$. No constraints on C yet!
 - For $n = 1$, $a_1 \leq C \cdot n \Leftrightarrow 2 \leq C$. Done. We have our first lower bound for C .

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- IH: Let $n \geq 2$. Then, assume $(\forall i \in \{0, 1, 2, \dots, n-1\}) [P(i)]$, where $P(i)$ means $a_i \leq C \cdot i$

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- IH: Let $n \geq 2$. Then, assume $(\forall i \in \{0, 1, 2, \dots, n-1\}) [P(i)]$, where $P(i)$ means $a_i \leq C \cdot i$
- IS: We attempt to prove $(P(0) \wedge P(1) \wedge P(2) \wedge \dots \wedge P(n-1)) \Rightarrow P(n)$:

$$\bigwedge_{i=0}^{i=n-1} (a_i \leq C \cdot i) \Rightarrow a_n \leq C \cdot n$$

Finding C

- IS: We attempt to prove $(P(1) \wedge P(2) \wedge \cdots \wedge P(n - 1)) \Rightarrow P(n)$:

$$\bigwedge_{i=0}^{i=n-1} (a_i \leq C \cdot i) \Rightarrow a_n \leq C \cdot n$$

- From the IH, and taking into consideration that $0 \leq \lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor \leq n$, we have (next slide):

Finding C

- From the IH, and taking into consideration that $0 \leq \lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor \leq n$, we have:

$$\begin{cases} a_{\lfloor n/4 \rfloor} \leq C \cdot \lfloor n/4 \rfloor \leq C \cdot \frac{n}{4} \\ a_{\lfloor n/2 \rfloor} \leq C \cdot \lfloor n/2 \rfloor \leq C \cdot \frac{n}{2} \end{cases}$$

- $a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/4 \rfloor} + 5n \leq C \cdot \frac{n}{2} + C \cdot \frac{n}{4} + 5n = \frac{n \cdot (3C + 20)}{4}$

Finding C

- We have:

$$a_n \leq \frac{n \cdot (3C + 20)}{4}$$

- We want:

$$a_n \leq C \cdot n$$

- Hence, we want a C such that:

$$\frac{n \cdot (3C + 20)}{4} \leq C \cdot n$$

Finding C

$$\begin{aligned} \frac{n(3C + 20)}{4} &\leq C \cdot n \stackrel{n \geq 1}{\Leftrightarrow} \\ \frac{(3C + 20)}{4} &\leq C \Leftrightarrow \\ 3C + 20 &\leq 4C \Leftrightarrow \\ C &\geq 20 \\ \Rightarrow C_{min} &= 20 \end{aligned}$$

Constraints

- From the IB: $C \geq 2$
- From the IS: $C \geq 20$
- Since we want to minimize C , we set $C = 20$.

STOP

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