The Emptier-Filler Game

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There will be a bin with numbers in it.

- If the bin is ever empty then E wins.
- If game goes forever and bin is always nonempty then F wins.

The Emptier-Filler Game on $\mathbb{N}$

1) F puts a **finite** multiset of $\mathbb{N}$ into the bin.
(e.g., bin has $\{1, 1, 1, 2, 3, 4, 9, 9, 18, 18\}$.

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(e.g., replace 18 with 99,999,999 17's and 5000 16's.)

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Which player has the winning strategy? What is that strategy?
WORK IN GROUPS!

Answer!

E wins!

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Strategy for E Keep removing the largest number in the box.

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The highest rank in the bin is r.

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The highest rank in the bin is r.

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After the first n moves of E the highest ranking number is $\leq r - 1$.

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This proof is **not rigorous**. We return to this point later.

The Emptier-Filler Game on Other Orderings

WORK IN GROUPS Determine who wins the Emptier-Filler Game on these orderings:

1. $\mathbb{Z}$
2. $\mathbb{Q}^{\geq 0}$
3. $\mathbb{Q}^{>0}$
4. $\mathbb{N} + \mathbb{N}$.
5. $\mathbb{N} + \mathbb{N} + \cdots + \mathbb{N}$ 100 times.
6. $\mathbb{N} + \mathbb{N} + \mathbb{N} + \cdots$ goes on forever

Answers!

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   Key When filler initially puts balls in the bin he only uses
   (say) the first 100,000 $\mathbb{N}$'s.
   At that point you are playing the game with $\mathbb{N}+\cdots+\mathbb{N}$
   (100,000 times).

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Question Let X be a set and $\preceq$ be an ordering on it. Let the $(X, \preceq)$-game be the game as above where we put elements of X in the bin.

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In the following sentence fill in the BLANK.
E can win the $(X, \preceq)$-game if and only if $(X, \preceq)$ BLANK.
WORK IN GROUPS!

Answer!

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Def $(X, \preceq)$ is **well ordered** if there are NO infinite decreasing sequences.

E can win the $(X, \preceq)$-game if and only if $(X, \preceq)$ is well ordered.

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Answer on next slide.

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From here, by the **IH**, E wins.
End of Proof

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Upshot You can do induction on any well ordered ordering.

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How to prove that? By an induction on a an even funkier ordering. We won't be doing that.

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Can E and F play so that F loses. Yes- F just never puts anything in the bin.