An Interesting Sum

Notation

Powerset

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This is standard and I thought I had said it, but I didn't so I am saying it now.

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We look at the sums of each set in 2^A.
$\mathrm{SUM}(\emptyset) = 0$
$\mathrm{SUM}(\{1\}) = 1$
$\mathrm{SUM}(\{4\}) = 4$
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$\mathrm{SUM}(\{1, 4\}) = 1 + 4 = 5$
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Could $\mathrm{SPS}(A)$ be of size 2^n? Work in groups to either

- find an A with $|\mathrm{SPS}(A)| = 2^n$, or
- show there is no such A.

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I leave this one for you.

How Small can $|\mathrm{SPS}(A)|$ Be?

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Work in groups to find $A \subseteq \mathbb{N}$ so that $\mathrm{SPS}(A)$ is small.

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Vote Yes, No, Unknown to Bill.

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  $|\mathrm{SPS}(1,\ldots,n)| = \frac{n(n+1)}{2} + 1$.
- Uses $n+1$. So the sumsets are of the form $n+1+x$ where
  $x \in \mathrm{SPS}(1,\ldots,n)$.

$\mathrm{SPS}(1, \ldots, n) = \{0, \ldots, \frac{n(n+1)}{2}\}$ (cont)

$$\mathrm{SPS}(1, \ldots, n+1) = \mathrm{SPS}(1, \ldots, n) \cup (\mathrm{SPS}(1, \ldots, n) + (n+1)).$$

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The second unionand is
$\{(n+1), 1+(n+1), \ldots, \frac{n(n+1)}{2} + (n+1)\} =$
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1) $\mathrm{SPS}(x_1, \ldots, x_n)$. By the IH there are $\geq \frac{n(n+1)}{2}$ of these sums.

2) $\mathrm{SUM}(B)$ where $B \subseteq \{x_1, \ldots, x_{n+1}\}$ and $x_{n+1} \in B$.
We show $\exists \geq n + 1$ of these sums that are not in $\mathrm{SPS}(x_1, \ldots, x_n)$.

For $A \subseteq \mathbb{N}$, $|\mathrm{SPS}(A)| \ge \frac{n(n+1)}{2} + 1$

IB $n = 1$. $A = \{x_1\}$. $\mathrm{SPS}(A) = \{0, x_1\}$. $|\mathrm{SPS}(A)| = 2 = \frac{1\times 2}{2} + 1$.
IH For all $x_1, \ldots, x_n \in \mathbb{N}$ $|\mathrm{SPS}(x_1, \ldots, x_n)| \ge \frac{n(n+1)}{2}$.
IS Want $(\forall x_1, \ldots, x_{n+1} \in \mathbb{N})$ $|\mathrm{SPS}(x_1, \ldots, x_{n+1})| \ge \frac{(n+1)(n+2)}{2}$.
We can assume $x_1 < \cdots < x_n < x_{n+1}$.

$\mathrm{SPS}(x_1, \ldots, x_{n+1})$ has two kinds of sets.

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Next Slide.

For $A \subseteq \mathbb{R}^{>0}$, $|\mathrm{SPS}(A)| \geq \frac{n(n+1)}{2} + 1$ (cont)

Recap

$x_1 < \cdots < x_{n+1}$.

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We need to show that there $n+1$ sums not in $\mathrm{SPS}(x_1, \ldots, x_n)$ and all differ from each other.

For $A \subseteq \mathbb{R}^{>0}$, $|\mathrm{SPS}(A)| \geq \frac{n(n+1)}{2} + 1$ (cont)

Key The largest number in $\mathrm{SPS}(x_1, \ldots, x_n)$ is $x_1 + \cdots + x_n$.

For $A \subseteq \mathbb{R}^{>0}$, $|\mathrm{SPS}(A)| \geq \frac{n(n+1)}{2} + 1$ (cont)

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$$x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_{n+1} > x_1 + \cdots + x_{i-1} + x_i + x_{i+1} + \cdots + x_n$$

$$x_{n+1} > x_i.$$

So all these $n+1$ new sums are $>$ than anything in $\mathrm{SPS}(x_1, \ldots, x_n)$.

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$$|\mathrm{SPS}(x_1, \ldots, x_{n+1})| \geq |\mathrm{SPS}(x_1, \ldots, x_n)| + n + 1$$

$$\geq \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}.$$

Other Domains

Recap What we Know Paying Attention to Domains

We have shown the following:

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This is two types of questions:

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2. Unusual domains: Primes, Powers-of-two.

Maximizing $|\mathrm{SPS}(A)|$

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1. The above is true for any X that contains powers of 2.
2. Work in Groups: Either find an X where the above is **False** or prove that, for all X, the above is **True**.

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FALSE if X is finite. Perhaps you don't want to count that.

TRUE If X is an infinite subset of $\mathbb{N}$. Take numbers that are space very far apart.
Proof that $|\mathrm{SPS}(X)| = 2^n$ is similar to proof that $|\mathrm{SPS}(2^0, \ldots, 2^{n-1})| = 2^n$.

Vote TRUE, FALSE, or UNKNOWN TO BILL on the following statement:
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Maximizing $|\mathrm{SPS}(A)|$

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TRUE but I won't prove it. Try on your own for $X = [0, 1]$.