Reciprocal Theorems

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All of them will be by induction.

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We may sometimes need the $n = 4$ base case:
$\frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{24} = 1$.

Proof One

IH+IS

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$$1 = \frac{1}{d_1} + \cdots + \frac{1}{d_n} = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-1}} + \frac{1}{d_n + 1} + \frac{1}{d_n(d_n + 1)}.$$

Proof Two. Bigger Base Case and $P(n) \rightarrow P(n+2)$

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For the case at hand we already did the $n = 3$ and $n = 4$ base case.

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Our next proof does this and make some other points of interest.

Proof Three. Load the IH

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Lets try to use it manually.

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It works so long as the last number is $\equiv 0 \pmod 2$.

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Loading the IH Proving a harder theorem so that the IH is stronger.

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Also NEED that the last number is $\equiv 0$. It is since $3d_n \equiv d_n \equiv 0$.

Proof Four. A Different Approach

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