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Logical Equivalence

CMSC250

Equivalences

- Let's observe the following truth table

p	q	$p \wedge q$	$q \wedge p$
F	***F***	***F***	***F***
F	***T***	***F***	***F***
T	***F***	***F***	***F***
T	***T***	***T***	***T***

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- **Conclusion:** $p \wedge q \equiv q \wedge p$

This symbol means
"logical equivalence"

An important equivalence

- Please fill – in the following truth table:

p	q	$p \implies q$	$(\sim p) \lor q$
F	***F***	***?***	***?***
F	***T***	***?***	***?***
T	***F***	***?***	***?***
T	***T***	***?***	***?***

$\Leftrightarrow$ VS $\equiv$

- $\Leftrightarrow$ *("if and only if")* is used to **form statements**, e.g
  - $p \Leftrightarrow (q \wedge (\sim r))$
- $\equiv$ *("logically equivalent to")* **compares two statements**, e.g
  - $(p \wedge q) \equiv (q \wedge p)$

Another important equivalence

- Let’s fill in the following truth table :

a	b	$\sim (a \wedge b)$	$(\sim a) \vee (\sim b)$
F	*F*	?	?
F	*T*	?	?
T	*F*	?	?
T	*T*	?	?

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- **These columns are the same!**
- **Conclusion:** $\sim (a \wedge b) \equiv (\sim a) \vee (\sim b)$

Another important equivalence

- Let's fill in the following truth table :

a	b	$\sim(a \land b)$	$(\sim a) \lor (\sim b)$
F	*F*	*T*	*T*
F	*T*	*T*	*T*
T	*F*	*T*	*T*
T	*T*	*F*	*F*

This result is known as
De Morgan's law

- **These columns are the same!**
- **Conclusion:** $\sim(a \land b) \equiv (\sim a) \lor (\sim b)$

Understanding De Morgan's Law

- $\sim(\text{"}Alice\ is\ Blonde\text{"} \wedge \text{"}Alice\ wears\ Green\ Dress\text{"})$: **Clearly true**

Understanding De Morgan's Law

- $\sim(\text{“}Alice\ is\ Blonde\text{”} \wedge \text{“}Alice\ wears\ Green\ Dress\text{”})$: **Clearly true**

- $(\sim\text{“}Alice\ is\ Blonde\text{”}) \vee (\sim\text{“}Alice\ wears\ Green\ Dress\text{”})$: **Also true!**

De Morgan’s Laws (there’s two of them)

$$\sim (a \vee b) \equiv (\sim a) \wedge (\sim b)$$

$$\sim (a \wedge b) \equiv (\sim a) \vee (\sim b)$$

- Conjunctions flipped to disjunctions, and vice versa
- Negation operator ($\sim$) distributed across terms
- These laws give us our first pair of equivalent expressions!

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- How do we prove an equivalence? (e.g $\sim(a \wedge b) \equiv (\sim a) \vee (\sim b)$)

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   - Can we do better?

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1. Truth tables
    - One major problem: for n variables, 2^n rows (input combinations) to enumerate!
    - Can we do better?
2. Laws of logical equivalence in a chain, one after the other!
    - We no longer have to compare 2^n input combinations to ensure that they all map to the same truth value (**T** or **F**). ☺
    - But somebody needs to code the system up!

Table of equivalences

Commutativity of binary operators	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associativity of binary operators	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributivity of binary operators	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws	$p \wedge T \equiv p$	$p \vee F \equiv p$
Negation laws	$p \vee ({\sim}p) \equiv T$	$p \wedge ({\sim}p) \equiv F$
Double negation	${\sim}({\sim}p) \equiv p$	
Idempotence	$p \wedge p \equiv p$	$p \vee p \equiv p$
De Morgan's axioms	${\sim}(p \wedge q) \equiv ({\sim}p) \vee ({\sim}q)$	${\sim}(p \vee q) \equiv ({\sim}p) \wedge ({\sim}q)$

- This exact table will be **given to you** during **all** exams where you might need it, so that you don't have to remember some "exotic" names

Table of equivalences

Universal bound laws	$p \vee T \equiv T$	$p \wedge F \equiv F$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of contradictions / tautologies	$\sim F \equiv T$	$\sim T \equiv F$
Contrapositive	$(a \Rightarrow b) \equiv ((\sim b) \Rightarrow (\sim a))$	
Equivalence between biconditional and implication	$a \Leftrightarrow b \equiv (a \Rightarrow b) \wedge (b \Rightarrow a)$	
Equivalence between implication and disjunction	$a \Rightarrow b \equiv \sim a \vee b$	

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Proving equivalences using laws

- Suppose we want to investigate if

$$(p \land r) \lor \big((p \lor s) \land (p \lor a)\big) \equiv p \lor (s \land a)$$

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- Suppose we want to investigate if

$$(p \wedge r) \vee \big((p \vee s) \wedge (p \vee a)\big) \equiv p \vee (s \wedge a)$$

- How many rows would the truth table have?
  - $2^4 = 16$ ☹ Too much time!

Proving equivalences using laws

- Suppose we want to investigate if

$$(p \wedge r) \vee \big((p \vee s) \wedge (p \vee a)\big) \equiv p \vee (s \wedge a\ )$$

- How many rows would the truth table have?
  - $2^4 = 16$ ☹ Too much time!
- Let’s see how we could use the laws of logical equivalence to prove this equivalence
  - Important: **document the laws!**

Solution

$$(p \wedge r) \vee \big((p \vee s) \wedge (p \vee a)\big)$$

$$\equiv (p \wedge r) \vee (p \vee (s \wedge a)) \qquad \textit{(Distributivity)}$$

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$$\equiv (p \wedge r) \vee (p \vee (s \wedge a\,)) \qquad \textit{(Distributivity)}$$

$$\equiv ((p \wedge r) \vee p) \vee (s \wedge a\,) \qquad \textit{(Associativity)}$$

Solution

$$(p \wedge r) \vee \big((p \vee s) \wedge (p \vee a)\big)$$

$$\equiv (p \wedge r) \vee (p \vee (s \wedge a\ ))$$ *(Distributivity)*

$$\equiv ((p \wedge r) \vee p) \vee (s \wedge a\ )$$ *(Associativity)*

$$\equiv (p \vee (p \wedge r)) \vee (s \wedge a\ )$$ *(Commutativity)*

Solution

$$(p \wedge r) \vee \big((p \vee s) \wedge (p \vee a)\big)$$

$\equiv (p \wedge r) \vee (p \vee (s \wedge a))$ *(Distributivity)*

$\equiv ((p \wedge r) \vee p) \vee (s \wedge a)$ *(Associativity)*

$\equiv (p \vee (p \wedge r)) \vee (s \wedge a)$ *(Commutativity)*

$\equiv p \vee (s \wedge a)$ *(Absorption)*

More equivalences

- Let's prove the following equivalences **true** or **false** together.

$$a \Rightarrow b \equiv (\sim b) \Rightarrow (\sim a) \quad \textit{(Contrapositive)}$$

$$a \Rightarrow b \equiv (\sim a) \Rightarrow (\sim b) \quad \textit{(Inverse Error)}$$

$$a \Leftrightarrow b \equiv \big((\sim a) \vee b\big) \wedge ((\sim b) \vee a)$$

Truth Tables vs Proofs of Equivalence

- When we want to show that $\phi(x_1, x_2, \ldots x_n) = \psi(x_1, x_2, \ldots, x_n)$:

Truth Table		Equivalence Proof	
Pro	**Con**	**Pro**	**Con**
Always works	Requires 2^n ***space***	Often occupies much less than 2^n space	There are some cases where it will still take 2^n space / time
No “cleverness” needed: just build all rows mechanically	Requires 2^n ***time***	Often spends much less than 2^n time	Requires “cleverness”

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