START

RECORDING

Discrete Probability Part 1

CMSC 250

Axiomatic Definitions, Basic Problems with Cards

Informal Definition of Probability

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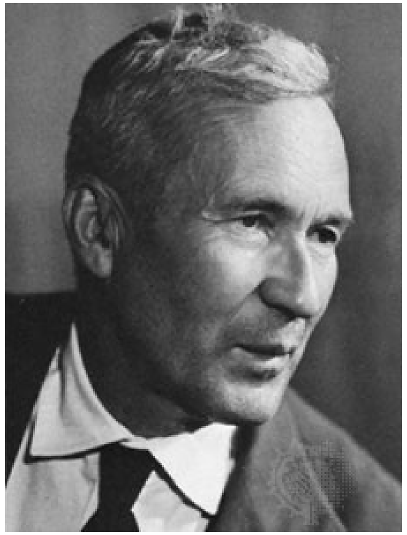
- This definition is owed to Andrey Kolmogorov, and assumes *that all possibilities are equally likely!*



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      - $\{HHH, HHT, HTH, HTT, THH, THT, TTH, \boldsymbol{TTT}\}$ (8 of them)
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Implicit assumption: all individual outcomes (HHH, HHT, HTH, ….) ***are considered equally likely (probability 1/8)***

Practice

- Experiment #2: I roll two dice.
  - Probability that I hit seven = ?

$\frac{1}{12}$	$\frac{1}{6}$	$\frac{7}{12}$	Something else

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      - $\{(1,1),(1,2),\ldots,(6,1)\}$ (36 of them)
    - Set of events where we hit 7.
      - $\{(2,5),(5,2),(3,4),\ (4,3),(1,6),\ (6,1)\}$ (6 of them)
    - Hence the answer: $\frac{6}{36} = \frac{1}{6}$

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    - Hence the answer: $\frac{6}{36}=\frac{1}{6}$
  - Probability that I hit two= ?
    - Same procedure

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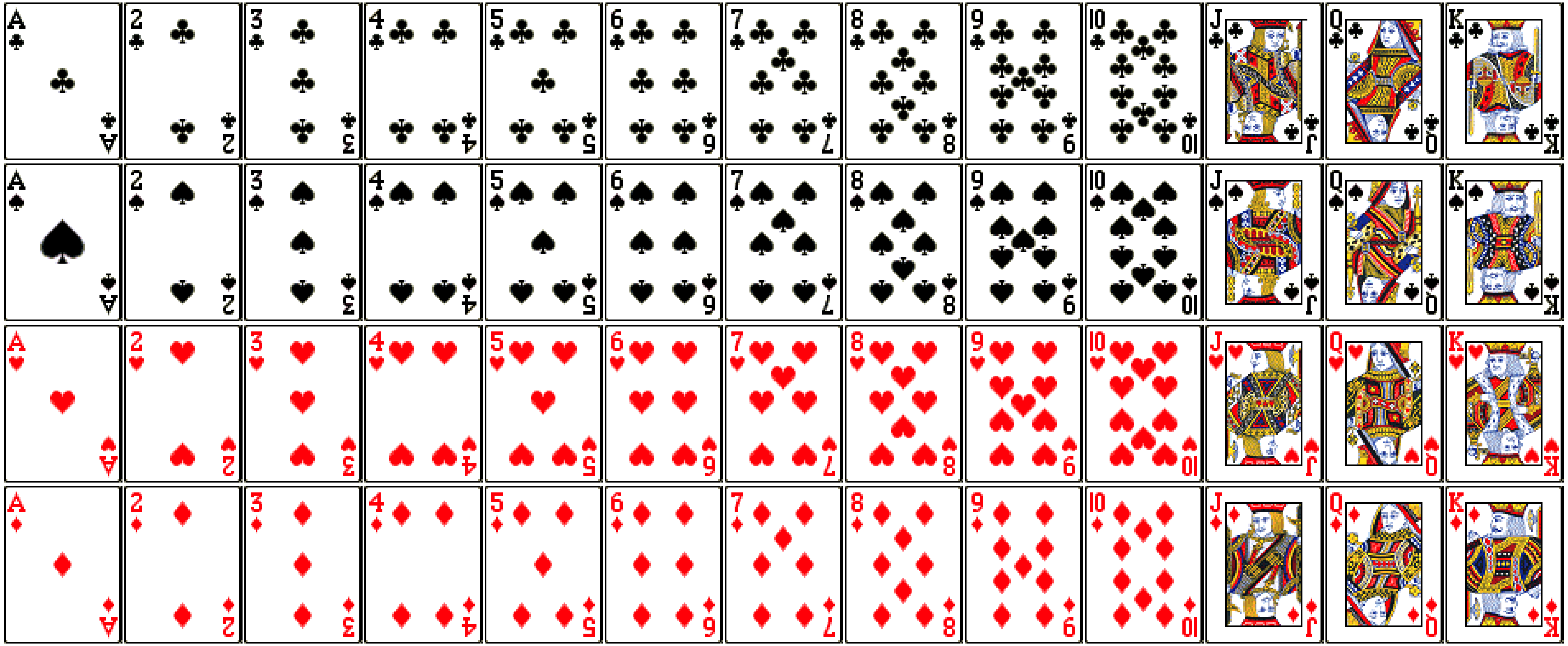
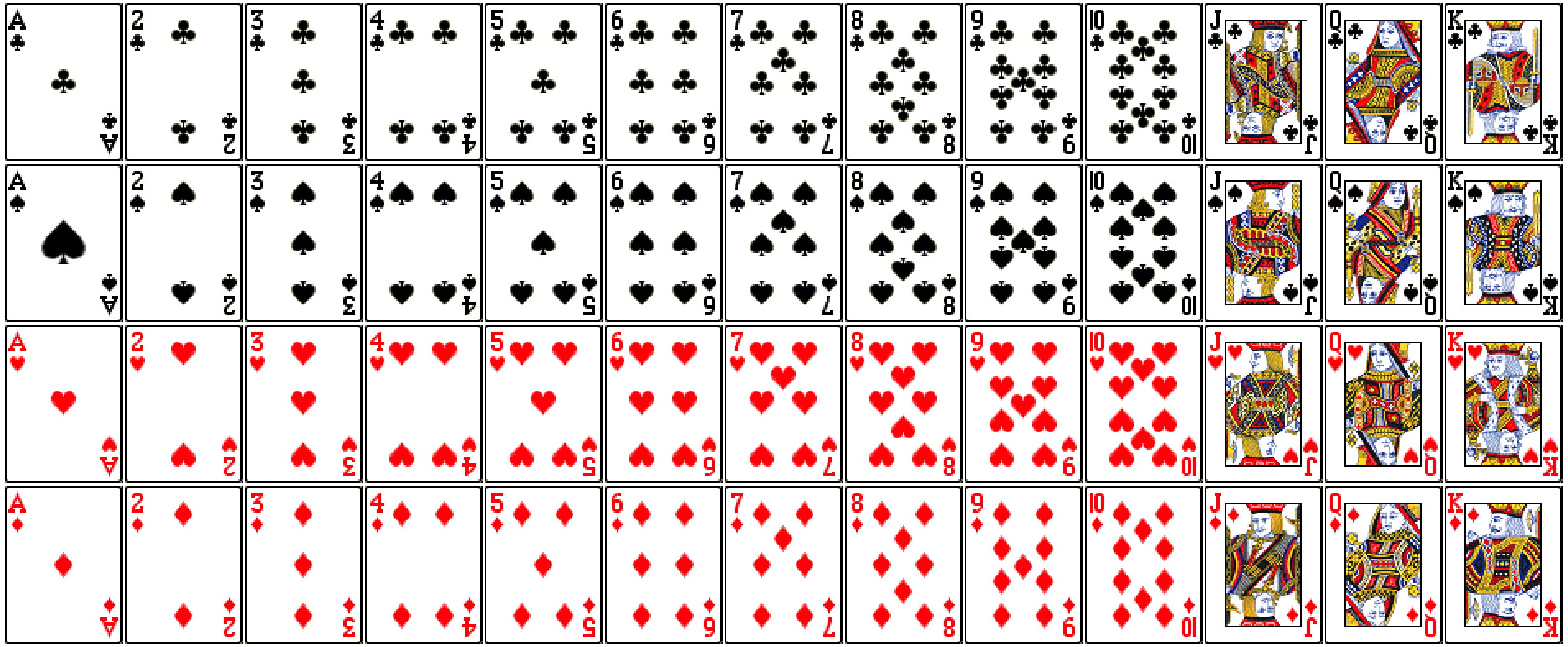
Poker Practice

- Full deck = 52 cards, 13 of each suit:



Poker Practice

- Full deck = 52 cards, 13 of each suit:
- **Flush**: 5 cards of the same suit
- What is the probability of getting a flush?



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- How likely is this?
  - Not at all likely: $\approx 0.002 = 0.2\%$ ☹

Likelihood of a Straight

- Straights are 5 cards of ***consecutive rank***
  - Ace can be ***either end*** (high or low)
  - ***No wrap-arounds*** (e.g Q K A 2 3, suits don't matter)

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That's $10 * 4^5$ ways. So, probability of a straight = $\frac{10*4^5}{\binom{52}{5}}$

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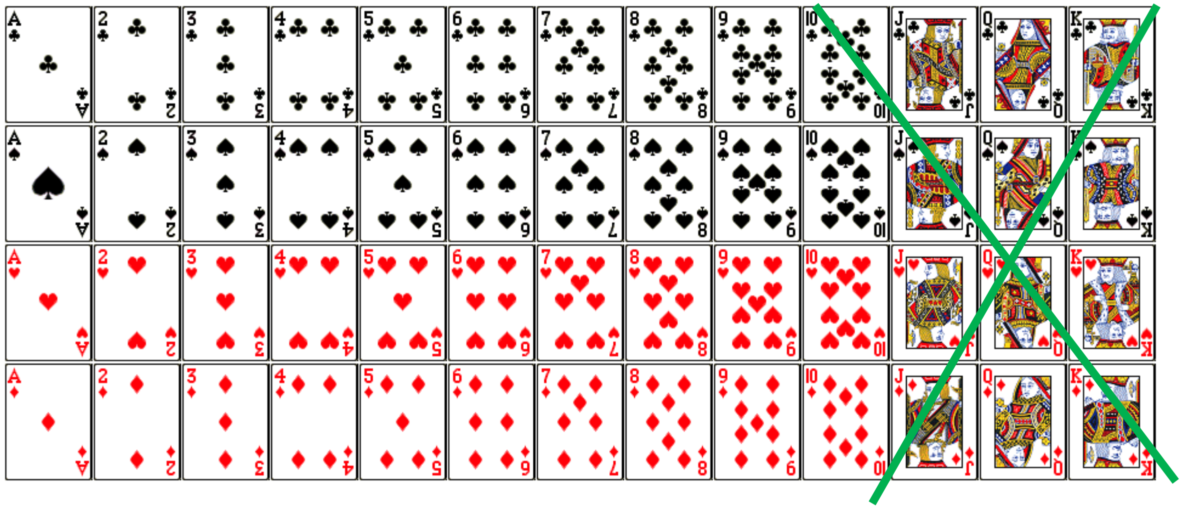
  - Hands like these are called straight flushes and Wikipedia does not include them.
  - ***How many straight flushes are there?***
  - 40. Here's why:
    - Pick rank: A through 10 (higher ranks don't allow straights) in 10 ways
    - Pick suit in 4 ways



Probability of Non-Straight Flush…

$$\frac{4 * \binom{13}{5} - 40}{\binom{52}{5}} = 0.001965$$

- This is how Wikipedia defines the probability of a flush. ☺

Probability of a Straight Flush...

$$\frac{40}{\binom{52}{5}} = 0.0000138517$$

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The expected # hands you need to play to get a straight flush is then

$$\left\lceil \frac{1}{0.0000138517} \right\rceil = 72{,}194$$

Same Caveat for Straights

- From the #straights we computed we will have to subtract the 40 possible straight flushes to get…

$$\frac{10 * 4^5 - 40}{\binom{52}{5}} = 0.003925$$

Same Caveat

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$$\frac{10*4^5-40}{\binom{52}{5}} = 0.003925 > 0.001965 = \text{probability of flush}$$

- *Flushes, being more rare, beat straights in poker.*

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  - The denominator will be $\binom{52}{5}$ (easy), so let's focus on the numerator:
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    2. Then, choose two of four suits in $\binom{4}{2} = 6$ ways.
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Severe overcount!

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- In the computation before, we included:
  - 2-of-a-kind
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- In the computation before, we included:
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  - 3-of-a-kind
  - 4-of-a-kind
  - Full House
- To properly compute, we would have to subtract **all** better hands possible with at least one pair.

Joint Probability

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  - $P(AB)$ (Perhaps most convenient, therefore most common)

Calculating Joints

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- Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$
- Probability that the first die is at most a 2 and the second one is 5 or 6
  - # outcomes of die roll is 6
  - # outcomes where first die is at most 2 is 2
    - Hence, probability of first die roll being at most 2 is $\frac{1}{3}$

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- Probability that the first die is at most a 2 and the second one is 5 or 6
  - # outcomes of die roll is 6
  - # outcomes where first die is at most 2 is 2
    - Hence, probability of first die roll being at most 2 is $\frac{1}{3}$
  - Similarly, probability of second die roll being 5 or 6 is $\frac{1}{3}$.
  - Hence, probability that **both** events happen (joint probability) is $\frac{1}{9}$.

Calculating Joints

- Jason's going to flip a coin and then pick a card from a 52-card deck.
  - Probability that the coin is heads and the card has rank 8?

$\frac{1}{2}$	$\frac{1}{26}$	$\frac{1}{32}$	Something else

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- This is because $P(coin = H) = \frac{1}{2}$ and $P(card\_rank = 8) = \frac{4}{52} = \frac{1}{13}$
  - So their joint probability is $\frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$

The Law of Joint Probability

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i)$$

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- Unfortunately, this “law” is not always applicable!
- It is applicable only when all the different events A_i are *independent* (sometimes called *marginally independent*) of each other.
- Let’s look at an example.

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  - ***NO!***
    - What is the probability that the die is even and the die is 2?

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$

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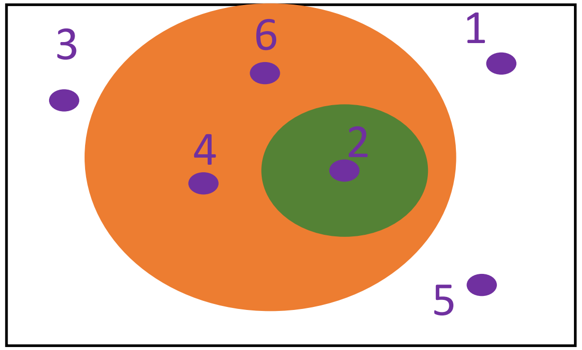
$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{5}$ $\frac{1}{6}$

Set-Theoretic Interpretation

- Notice that the event A: "Die roll is even" is a superset of the event B: "Die roll comes 2"



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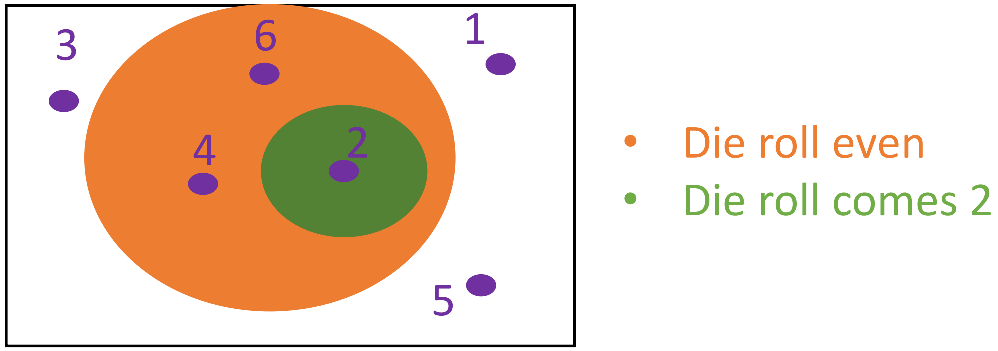

- Since $A \cap B = A$, $P(A \cap B) = P(A) = \frac{1}{6}$

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  - Clearly, it **can't** be

(probability Jason gets an A) X (probability Jason gets a B) = $\frac{\mathbf{1}}{\mathbf{7}} \times \frac{\mathbf{1}}{\mathbf{7}} = \frac{\mathbf{1}}{\mathbf{49}}$

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  - Events such as these are called *disjoint* or *mutually disjoint.*

Set-Theoretic Interpretation

- A = “Jason gets an A in USND’s 250”
- G=“Jason gets a G in USND’s 250”

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- Note that $A \cap G = \emptyset$, so there are no common outcomes.
  - So $P(A \cap G) = 0$

Calculating Joints

- I have my original die again.
  - Probability that it comes up 1, 2 or 3 = $\frac{1}{2}$
  - Probability that it comes up 3, 4 or 5 = $\frac{1}{2}$
  - What is the probability that it comes up 1, 2 or 3 **and** 3, 4 or 5?

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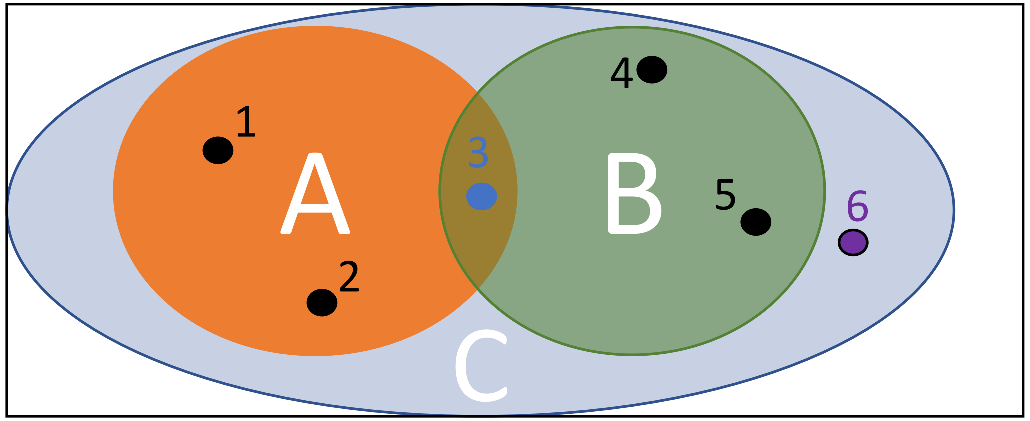
$\frac{1}{6}$ $\frac{1}{5}$ $\frac{1}{4}$ $\frac{1}{3}$

  - Note that the only common outcome between the two events is **3**, which can come up only **once** out of **six** possibilities.

Set-Theoretic Interpretation

- Let A = dice comes up 1, 2, or 3
- Let B = dice comes up 3, 4, or 5
- Let C = dice comes up 1, 2, 3, 4, 5 OR 6



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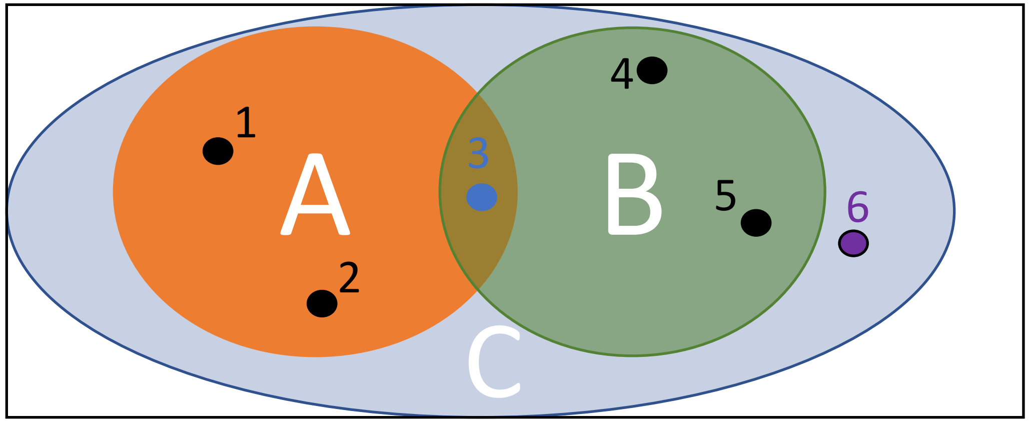

- Then, probability that the dice comes up 3 = $\frac{1}{6}$

STOP RECORDING