Homework 5

250H Spr 2024

Let p be a prime. Show that $\sqrt{p} \notin \mathbf{Q}$ using Unique Factorization.

Proof: For the sake of contradiction, assume that $\sqrt{p} \in \mathbf{Q}$. Then by definition, $\sqrt{p} = a / b$ for $a, b \in \mathbf{Z}$ and

$b \neq 0$. Let us add the restriction that a and b have no common factors. So,

$$\sqrt{p} = a / b$$

$$p = a^2 / b^2$$

$$b^2 p = a^2$$

Let p be a prime. Show that $\sqrt{p} \notin \mathbf{Q}$ using Unique Factorization.

Let us factor a and b into primes so $a = p_1^{a1} \times \cdots \times p_m^{am}$ and $b = p_1^{b1} \times \cdots \times p_m^{bm}$. WLOG let $p = p_1$. So,

$$(p_1^{b1} \times \cdots \times p_m^{bm})^2 p = (p_1^{a1} \times \cdots \times p_m^{am})^2$$

$$(p_1^{2b1} \times \cdots \times p_m^{2bm}) p = p_1^{2a1} \times \cdots \times p_m^{2am}$$

$$p_1^{2b1+1} \times \cdots \times p_m^{2bm} = p_1^{2a1} \times \cdots \times p_m^{2am}$$

$$p_1^{2b1+1} = p_1^{2a1}$$

So, this would have to mean $2b_1 + 1 = 2a_1$. This cannot happen as then that would mean by definition $2b_1 + 1$ is odd and $2a_1$ is even. Since, an odd cannot equal an even, we have a contradiction. ☽

Let p be a prime. Show that $p^{1/c} \notin \mathbf{Q}$ using Unique Factorization.

Proof: For the sake of contradiction, assume that $p^{1/c} \in \mathbf{Q}$. Then by definition, $p^{1/c} =$ a / b for a, b $\in \mathbf{Z}$ and
b ≠ 0. Let us add the restriction that a and b have no common factors. So,

$$p^{1/c} = a / b$$

$$p = a^c / b^c$$

$$b^c p = a^c$$

Let p be a prime. Show that $p^{1/c} \notin \mathbf{Q}$ using Unique Factorization.

Let us factor a and b into primes so $a = p_1^{a1} \times \cdots \times p_m^{am}$ and $b = p_1^{b1} \times \cdots \times p_m^{bm}$. WLOG let $p = p_1$. So,

$$(p_1^{b1} \times \cdots \times p_m^{bm})^c p = (p_1^{a1} \times \cdots \times p_m^{am})^c$$

$$(p_1^{cb1} \times \cdots \times p_m^{cbm}) p = p_1^{ca1} \times \cdots \times p_m^{cam}$$

$$p_1^{cb1+1} \times \cdots \times p_m^{cbm} = p_1^{ca1} \times \cdots \times p_m^{cam}$$

$$p_1^{cb1+1} = p_1^{ca1}$$

So, this would have to mean $cb_1 + 1 = ca_1$. The LHS $\equiv 1 \pmod{c}$ and RHS $\equiv 0 \pmod{c}$. Since this cannot happen, we have a contradiction. ☽