How to Write Proofs

250H

What is the point of a proof?

- Prove that a statement is true clearly and without **ambiguity**

Types of Proofs

- **Direct**
    - p ➔ q
    - Assume p
    - Show q

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T	T	T
T	F	F
F	T	T
F	F	T

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- **Contradiction**
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  - Assume p and ¬q
  - Show something goes wrong

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Types of Proofs

- **Direct**
  - $p \rightarrow q$
  - Assume p
  - Show q
- **Contradiction**
  - $p \rightarrow \neg q$
  - Assume p and $\neg q$
  - Show something goes wrong
- **Contrapositive**
  - $\neg q \rightarrow \neg p$
  - Assume $\neg q$
  - Show $\neg p$

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Tips on how to start a proof

- What do we know
- What do we want to show
- What definitions might we need
- What type of proof are we going to use

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- What type of proof are we going to use
  - Direct? No
  - Contradiction? Possibly
  - Contrapositive? Possibly

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- Do the algebra:
  - Then, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

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- Finish it:
  - Thus, if n^2 is even, then n is even.

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Let $n \in \mathbb{Z}$. For the sake of contradiction, assume n^2 is even and n is odd. If n is odd then $n = 2k+1$ where k is an integer by the definition of an odd number. Then,

$$n^2 = (2k+1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1.$$

Hence we have a contradiction as $2(2k^2 + 2k) + 1$ is odd since $2k^2 + 2k$ is an integer. Thus, if n^2 is even, then n is even. ☽

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  - Then, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
- Spell out your result:
  - Hence $2(2k^2 + 2k) + 1$ is odd since $2k^2 + 2k$ is an integer.
- Finish it:
  - So, if n is odd, then n^2 is odd.
  - Thus, if n^2 is even, then n is even.

Example: Let $n \in Z$. Prove that if n^2 is even, then n is even.

Proof:

Let $n \in Z$. Assume by way of contrapositive that n is odd.If n is odd then $n = 2k+1$ where k is an integer by the definition of an odd number. Then,

$$n^2 = (2k+1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1.$$

Hence, $2(2k^2 + 2k) + 1$ is odd since $2k^2 + 2k$ is an integer. So, if n is odd, then n^2 is odd. Thus, if n^2 is even, then n is even. ☽

Prove: The sum of two numbers with the same parity is always even.

Proof: Let x and y be integers. Consider the cases:

Case 1: x and y are even. If x and y are even, by definition $x = 2k$ and $y = 2h$ were k and h are integers. So,

$$x + y = 2k + 2h = 2 (k + h)$$

Since $k + h$ is an integer, $x + y$ is even.

Case 2: x and y are odd. If x and y are odd, by definition $x = 2k + 1$ and $y = 2h + 1$ were k and h are integers. So,

$$x + y = 2k + 2h +2 = 2 (k + h + 1)$$

Since $k + h + 1$ is an integer, $x + y$ is even.

So, our statement holds. ☽

Tips

- Do not assume your reader knows all definitions
- Do not assume your reader sees what you see
    - It is clear that blah blah blah
    - No it is not
- Do not make things complicated for your reader. It does not make you look more intelligent.

What did we prove?

Proof: We consider two cases.

Case 1. a and b are even. Then $a = 2r$ and $b = 2s$ for integers r and s. Thus, $a^2 - b^2 = (2r)^2 - (2s)^2 = 4r^2 - 4s^2 = 2(2r^2 - 2s^2)$. Since $2r^2 - 2s^2$ is an integer, $a^2 - b^2$ is even.

Case 2. a and b are odd. Then $a = 2r + 1$ and $b = 2s + 1$ for integers r and s. Thus, $a^2 - b^2 = (2r + 1)^2 - (2s + 1)^2 = (4r^2 + 4r + 1) - (4s^2 + 4s + 1) = 4r^2 + 4r - 4s^2 - 4s = 2(2r^2 + 2r - 2s^2 - 2s)$. Since $2r^2 + 2r - 2s^2 - 2s$ is an integer, $a^2 - b^2$ is even.

Statement: If a and b are integers with the same parity, then $a^2 - b^2$ is even.

What did we prove?

Proof: Assume that x is even. Then x = 2a for some integer a. So,

$$3x^2 - 4x - 5 = 3(2a)^2 - 4(2a) - 5 = 12a^2 - 8a - 5 = 2(6a^2 - 4a - 3) + 1$$

Since $6a^2 - 4a - 3$ is an integer, $3x^2 - 4x - 5$ is odd.

For the converse, assume that x is odd. So, $x = 2b + 1$, where $b \in Z$. Therefore,

$$3x^2 - 4x - 5 = 3(2b + 1)^2 - 4(2b + 1) - 5 = 3(4b^2 + 4b + 1) - 8b - 4 - 5$$

$$= 12b^2 + 4b - 6 = 2(6b^2 + 2b - 3)$$

Since $6b^2 + 2b - 3$ is an integer, $3x^2 - 4x - 5$ is even.

Statement: If x is an integer, then $3x^2 - 4x - 5$ is the opposite parity of x.

The following is an attempted proof of a result. What is the result and is the attempted proof correct?

Proof: Assume, without loss of generality, that x is even. Then $x = 2a$ for some integer a. Thus,

$$xy^2 = (2a)y^2 = 2(ay^2)$$

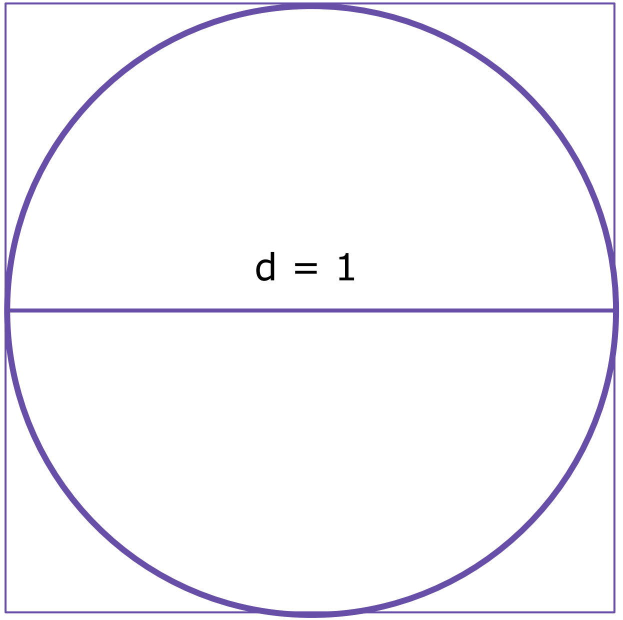
Since ay^2 is an integer, xy^2 is even.

Statement: If x is even or y is even are integers, then xy^2 is even.

Correct: No. Without loss of generality can not be used here.

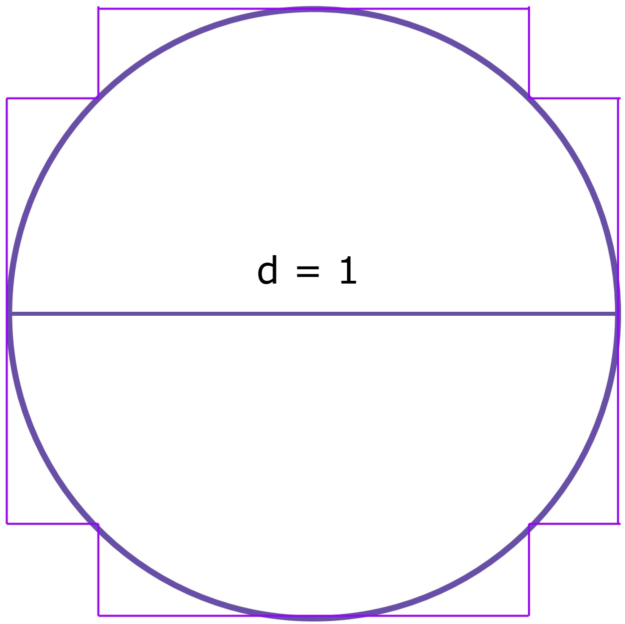
Proof?: π = 4

Proof: Consider a square with a perimeter of 4 and a circle inside that square with a diameter of 1.



Proof?: π = 4

Proof: Invert the Corners. The perimeter is still 4



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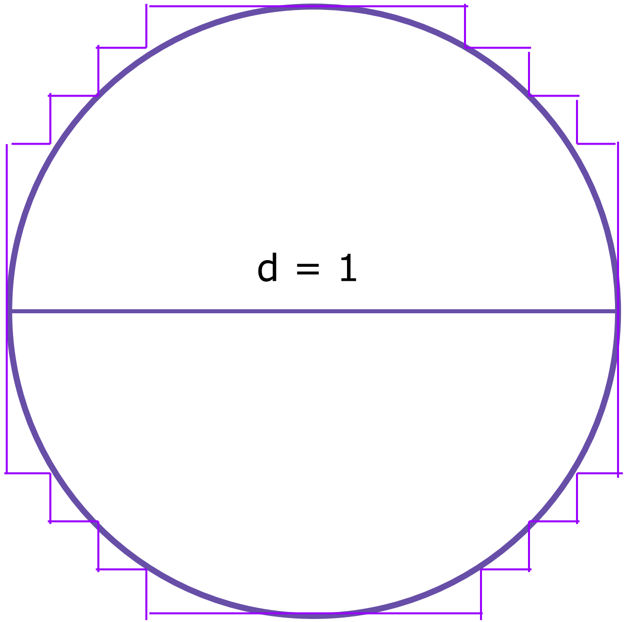
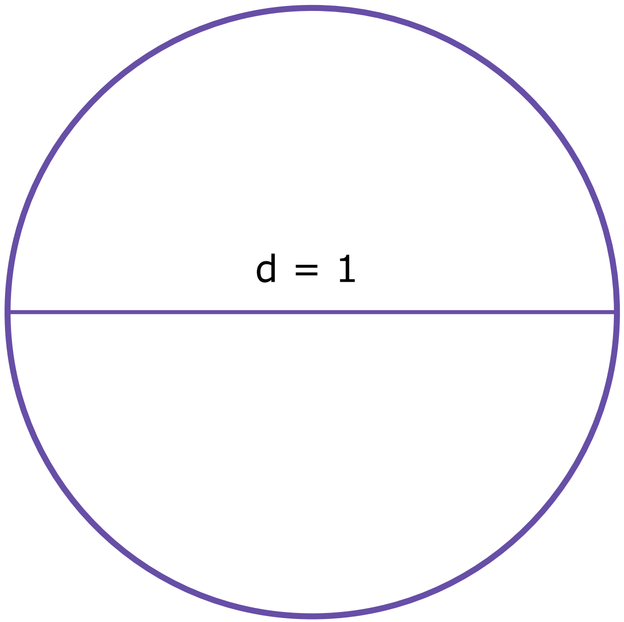
Proof: Invert the Corners again. The perimeter is still 4



Proof?: $\pi = 4$

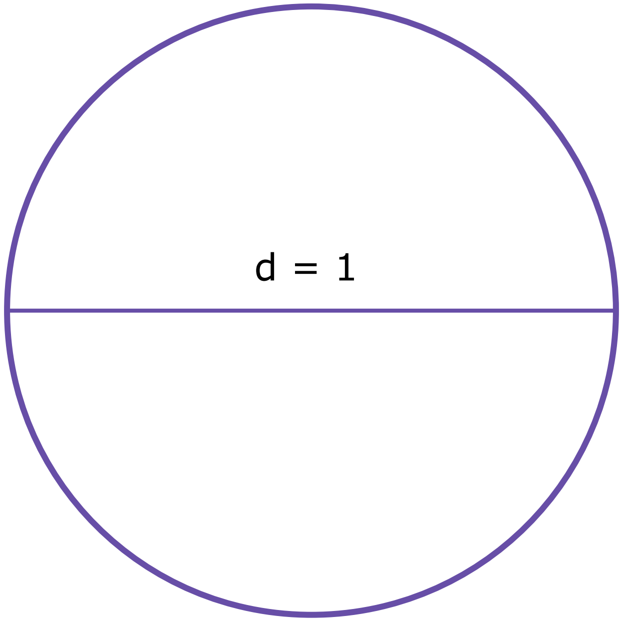
Proof: Repeat infinity times. We know $C = 2\pi r$. Here $r = ½$. So, $C = 2\pi(½) = \pi = 4$.



Proof?: π = 4

Proof: Repeat infinity times. We know $C = 2\pi r$. Here $r = \frac{1}{2}$. So, $C = 2\pi(\frac{1}{2}) = \pi = 4$.

Why is this wrong?



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Why is this wrong? We never actually exactly match the circle. Pictures are not proofs, they can only help understanding.