Ramsey's Theorem on Graphs

Exposition by William Gasarch

1 Introduction

Imagine that you have 6 people at a party. We assume that, for every pair of them, either THEY KNOW EACH OTHER or NEITHER OF THEM KNOWS THE OTHER. So we are assuming that if x knows y, then y knows x.

Claim: Either there are at least 3 people *all of whom know one another*, or there are at least 3 people *no two of whom know each other* (or both).

Proof of Claim:

Let the people be $p_1, p_2, p_3, p_4, p_5, p_6$. Now consider p_6.

Among the other 5 people, either there are at least 3 people that p_6 knows, or there are at least 3 people that p_6 does not know.

Why is this?

Well, suppose that, among the other 5 people, there are at most 2 people that p_6 knows, and at most 2 people that p_6 does not know. Then there are only 4 people other than p_6, which contradicts the fact that there are 5 people other than p_6.

Suppose that p_6 knows at least 3 of the others. We consider the case where p_6 knows p_1, p_2, and p_3. All the other cases are similar.

If p_1 knows p_2, then p_1, p_2, and p_6 all know one another. HOORAY!

If p_1 knows p_3, then p_1, p_3, and p_6 all know one another. HOORAY!

If p_2 knows p_3, then p_2, p_3, and p_6 all know one another. HOORAY!

What if *none* of these scenarios holds? Then none of these three people (p_1, p_2, p_3) knows either of the other 2. HOORAY!

Suppose now that p_6 does not know at least 3 of the others. We consider the case where p_6 does not know p_1, p_2, and p_3. All the other cases are similar.

If p_1 does not know p_2, then p_1, p_2, and p_6 all do not know one another. HOORAY!

If p_1 does not know p_3, then p_1, p_3, and p_6 all do not know one another. HOORAY!

If p_2 does not know p_3, then p_2, p_3, and p_6 all do not know one another. HOORAY!

What if *none* of these scenarios holds? Then all of these three people (p_1, p_2, p_3) knows the other 2. HOORAY! *End of Proof of Claim*

We want to generalize this observation.

Notation 1.1 N is the set of all positive integers. If $n \in \mathsf{N}$, then $[n]$ is the set $\{1, \ldots, n\}$.

Def 1.2 A *graph* G consists of a set V of vertices and a set E of edges. The edges are *unordered* pairs of vertices.

Note 1.3 In general, a graph can have an edge $\{i, j\}$ with $i = j$. Here, however, every edge of a graph is an unordered pair of *distinct* vertices (i.e., an unordered pair $\{i, j\}$ with $i \neq j$).

Def 1.4 Let $c \in \mathsf{N}$. Let $G = (V, E)$ be a graph. A *c-coloring of the edges of G* is a function $COL : E \rightarrow [c]$. Note that there are no restrictions on COL.

Note 1.5 In the Graph Theory literature there are (at least) two kinds of coloring. We present them in this note so that if you happen to read the literature and they are using coloring in a different way then in these notes, you will not panic.

- Vertex Coloring. Usually one says that the vertices of a graph are c-colorable if there is a way to assign each vertex a color, using no more than c colors, such that no two adjacent vertices (vertices connected by an edge) are the same color. Theorems are often of the form 'if a graph G has property BLAH BLAH then G is c-colorable' where they mean vertex c-colorable. We **will not** be considering these kinds of colorings.

- Edge Colorings. Usually this is used in the context of Ramsey Theory and Ramsey-type theorems. Theorems begin with 'for all c-coloring of a graph G BLAH BLAH happens' We **will** be considering these kinds of colorings.

Def 1.6 Let $n \in \mathsf{N}$. The *complete graph on n vertices*, denoted K_n, is the graph

$$\begin{aligned} V = & \ [n] \\ E = & \ \{\{i,j\} \mid i,j \in [n]\} \end{aligned}$$

Example 1.7 Let G be the complete graph on 10 vertices. Recall that the vertices are $\{1,\ldots,10\}$. We give a 3-coloring of the edges of G:

$$COL(\{x,y\}) = \begin{cases} 1 & \text{if } x+y \equiv 1 \pmod 3; \\ 2 & \text{if } x+y \equiv 2 \pmod 3; \\ 3 & \text{if } x+y \equiv 0 \pmod 3. \end{cases}$$

Lets go back to our party! We can think of the 6 people as vertices of K_6. We can color edge $\{i,j\}$ RED if i and j know each other, and BLUE if they do not.

Def 1.8 Let $G = (V,E)$ be a graph, and let COL be a coloring of the edges of G. A set of edges of G is *monochromatic* if they are all the same color.

Let $n \geq 2$. Then *G has a monochromatic K_n* if there is a set V' of n vertices (in V) such that

- there is an edge between every pair of vertices in V': $\{\{i,j\} \mid i,j \in V'\} \subseteq E$
- all the edges between vertices in V' are the same color: there is some $l \in [c]$ such that $COL(\{i,j\}) = l$ for all $i,j \in V'$. (I.e., V' is monochromatic.)

We now restate our 6-people-at-a-party theorem:

Theorem 1.9 *Every 2-coloring of the edges of K_6 has a monochromatic K_3.*

2 The Full Theorem

From the last section, we know the following:

If you want an n such that you get a monochromatic K_3 no matter how you 2-color K_n, then $n = 6$ will suffice.

What if you want to guarantee that there is a monochromatic K_4? What if you want to use 17 colors?

The following is known as *Ramsey's Theorem*. It was first proved in [3] (see also [1], [2]).

For all $c, m \geq 2$, there exists $n \geq m$ such that every c-coloring of K_n has a monochromatic K_m.

We will provide several proofs of this theorem for the $c = 2$ case. We will assume the colors are RED and BLUE (rather than the numbers 1 and 2). The general-c case (where c can be *any* integer $i \geq 2$) and other generalizations may show up on homework assignments.

3 Proof of Ramsey's Theorem

Theorem 3.1 *For every $m \geq 2$, $R(m)$ exists and $R(m) \leq 2^{2m-2}$.*

Proof:

Let COL be a 2-coloring of $K_{2^{2m-2}}$. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{2m-1},$$

and a sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots, V_{2m-1},$$

that are based on COL.

Here is the intuition: Vertex $x_1 = 1$ has $2^{2m-2} - 1$ edges coming out of it. Some are RED, and some are BLUE. Hence there are at least 2^{2m-3} RED edges coming out of x_1, or there are at least 2^{2m-3} BLUE edges coming out of x_1. To see this, suppose, by way of contradiction, that it is false, and let N.E. be the total number of edges coming out of x_1. Then

$$\text{N.E.} \leq (2^{2m-3} - 1) + (2^{2m-3} - 1) = (2 \cdot 2^{2m-3}) - 2 = 2^{2m-2} - 2 < 2^{2m-2} - 1$$

Let c_1 be a color such that x_1 has at least 2^{2m-3} edges coming out of it that are colored c_1. Let V_1 be the set of vertices v such that $COL(\{v, x_1\}) = c_1$. Then keep iterating this process.

We now describe it formally.

$$
\begin{aligned}
V_0 = &\ [2^{2m-2}] \\
x_1 = &\ 1
\end{aligned}
$$

$$
c_1 = \begin{cases} \text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \geq 2^{2m-3} \\ \text{BLUE} & \text{otherwise} \end{cases}
$$

$$
V_1 = \{v \in V_0 \mid COL(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \geq 2^{2m-3})
$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$
x_i = \text{ the least number in } V_{i-1}
$$

$$
c_i = \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_i\}) = \text{RED}\}| \geq 2^{(2m-2)-i}; \\ \text{BLUE} & \text{otherwise.} \end{cases}
$$

$$
V_i = \{v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i\} \text{ (note that } |V_i| \geq 2^{(2m-2)-i})
$$

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. Note that

$$
|V_{2m-2}| \geq 2^{(2m-2)-(2m-2)} = 2^0 = 1
$$

Thus if $i-1 = 2m-2$ (equivalently, $i = 2m-1$), then $V_{i-1} = V_{2m-2} \neq \emptyset$, but there is no guarantee that V_i $(= V_{2m-1})$ is nonempty. Hence we can define

$$
x_1, \ldots, x_{2m-1}
$$

Consider the colors

$$
c_1, c_2, \ldots, c_{2m-2}
$$

Each of these is either RED or BLUE. Hence there must be at least $m-1$ of them that are the same color. Let $i_1, \ldots, i_{m-1}$ be such that $i_1 < \cdots < i_{m-1}$ and

$$
c_{i_1} = c_{i_2} = \cdots = c_{i_{m-1}}
$$

Denote this color by c, and consider the m vertices

$$
x_{i_1}, x_{i_2}, \cdots, x_{i_{m-1}}, x_{i_{m-1}+1}
$$

To see why we have listed m vertices but only $m-1$ colors, picture the following scenario: You are building a fence row, and you want (say) 7 sections of fence. To do that, you need 8 fence posts to hold it up. Now think of the fence posts as vertices, and the sections of fence as edges between successive vertices, and recall that every edge has a color associated with it.

Claim: The m vertices listed above form a monochromatic K_m.
Proof of Claim:

First, consider vertex x_{i_1}. The vertices

$$x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_{m-1}+1}$$

are elements of V_{i_1}, hence the edges

$$\{x_{i_1}, x_{i_2}\}, \ldots \{x_{i_1}, x_{i_{m-1}}\}, \{x_{i_1}, x_{i_{m-1}+1}\}$$

are colored with c_{i_1} $(= c)$.

Then consider each of the remaining vertices in turn, starting with vertex x_{i_2}. For example, the vertices

$$x_{i_3}, \ldots, x_{i_{m-1}}, x_{i_{m-1}+1}$$

are elements of V_{i_2}, hence the edges

$$\{x_{i_2}, x_{i_3}\}, \ldots \{x_{i_2}, x_{i_{m-1}}\}, \{x_{i_2}, x_{i_{m-1}+1}\}$$

are colored with c_{i_2} $(= c)$.
End of Proof of Claim ▮

4 Proof of the Infinite Ramsey Theorem

We now consider infinite graphs.

Notation 4.1 K_{N} is the graph (V, E) where

$$\begin{aligned} V =&\ \ \mathsf{N} \\ E =&\ \ \{\{x, y\} \mid x, y \in \mathsf{N}\} \end{aligned}$$

Def 4.2 Let $G = (V, E)$ be a graph with $V = \mathsf{N}$, and let COL be a coloring of the edges of G. A set of edges of G is *monochromatic* if they are all the same color (this is the same as for a finite graph).

G has a monochromatic K_{N} if there is an infinite set V' of vertices (in V) such that

- there is an edge between every pair of vertices in V'
- all the edges between vertices in V' are the same color

Theorem 4.3 *Every 2-coloring of the edges of K_{N} has a monochromatic K_{N}.*

Proof:

(Note: this proof is similar to the proof of Theorem 3.1.)

Let COL be a 2-coloring of K_{N}. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on COL.

Here is the intuition: Vertex $x_1 = 1$ has an infinite number of edges coming out of it. Some are RED, and some are BLUE. Hence there are an infinite number of RED edges coming out of x_1, or there are an infinite number of BLUE edges coming out of x_1 (or both). Let c_1 be a color such that x_1 has an infinite number of edges coming out of it that are colored c_1. Let V_1 be the set of vertices v such that $COL(\{v, x_1\}) = c_1$. Then keep iterating this process.

We now describe it formally.

$$V_0 = \mathsf{N}$$
$$x_1 = 1$$

$$c_1 = \begin{cases} \text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$V_1 = \{v \in V_0 \mid COL(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \text{ is infinite)}$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$x_i = \quad \text{the least number in } V_{i-1}$$

$$c_i = \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_i\}) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$V_i = \quad \{v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i\} \text{ (note that } |V_i| \text{ is infinite)}$$

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. We an show by induction that, for every i, V_i is infinite. Hence the sequence

$$x_1, x_2, \ldots,$$

is infinite.

Consider the infinite sequence

$$c_1, c_2, \ldots$$

Each of the colors in this sequence is either RED or BLUE. Hence there must be an infinite sequence $i_1, i_2, \ldots$ such that $i_1 < i_2 < \cdots$ and

$$c_{i_1} = c_{i_2} = \cdots$$

Denote this color by c, and consider the vertices

$$x_{i_1}, x_{i_2}, \cdots$$

Using an argument similar to the one we used in the proof of Theorem 3.1 (to show that we had a monochromatic K_m), we can show that these vertices form a monochromatic K_{N}. ∎

References

[1] R. Graham, A. Rothchild, and J. Spencer. *Ramsey Theory*. Wiley, 1990.

[2] B. Landmann and A. Robertson. *Ramsey Theory over the Integers*. AMS, 2003.

[3] F. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, 30:264–286, 1930. Series 2.