Ramsey’s Theorem on Graphs

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1 Introduction

Imagine that you have 6 people at a party. We assume that, for every pair of them, either THEY KNOW EACH OTHER or NEITHER OF THEM KNOWS THE OTHER. So we are assuming that if $x$ knows $y$, then $y$ knows $x$.

**Claim:** Either there are at least 3 people all of whom know one another, or there are at least 3 people no two of whom know each other (or both).

**Proof of Claim:**

Let the people be $p_1, p_2, p_3, p_4, p_5, p_6$. Now consider $p_6$.

*Among the other 5 people, either there are at least 3 people that $p_6$ knows, or there are at least 3 people that $p_6$ does not know.*

Why is this?

Well, suppose that, among the other 5 people, there are at most 2 people that $p_6$ knows, and at most 2 people that $p_6$ does not know. Then there are only 4 people other than $p_6$, which contradicts the fact that there are 5 people other than $p_6$.

Suppose that $p_6$ knows at least 3 of the others. We consider the case where $p_6$ knows $p_1, p_2, and p_3$. All the other cases are similar.

If $p_1$ knows $p_2$, then $p_1, p_2, and p_6$ all know one another. HOORAY!

If $p_1$ knows $p_3$, then $p_1, p_3, and p_6$ all know one another. HOORAY!

If $p_2$ knows $p_3$, then $p_2, p_3, and p_6$ all know one another. HOORAY!

What if *none* of these scenarios holds? Then none of these three people $(p_1, p_2, p_3)$ knows either of the other 2. HOORAY!

Suppose now that $p_6$ does not know at least 3 of the others. We consider the case where $p_6$ does not know $p_1, p_2, and p_3$. All the other cases are similar.

If $p_1$ does not know $p_2$, then $p_1, p_2, and p_6$ all do not know one another. HOORAY!

If $p_1$ does not know $p_3$, then $p_1, p_3, and p_6$ all do not know one another. HOORAY!
If $p_2$ does not know $p_3$, then $p_2$, $p_3$, and $p_6$ all do not know one another. HOORAY!

What if none of these scenarios holds? Then all of these three people ($p_1$, $p_2$, $p_3$) knows the other 2. HOORAY! End of Proof of Claim

We want to generalize this observation.

**Notation 1.1** $\mathbb{N}$ is the set of all positive integers. If $n \in \mathbb{N}$, then $[n]$ is the set $\{1, \ldots, n\}$.

**Def 1.2** A graph $G$ consists of a set $V$ of vertices and a set $E$ of edges. The edges are unordered pairs of vertices.

**Note 1.3** In general, a graph can have an edge $\{i, j\}$ with $i = j$. Here, however, every edge of a graph is an unordered pair of distinct vertices (i.e., an unordered pair $\{i, j\}$ with $i \neq j$).

**Def 1.4** Let $c \in \mathbb{N}$. Let $G = (V, E)$ be a graph. A $c$-coloring of the edges of $G$ is a function $COL : E \rightarrow [c]$. Note that there are no restrictions on $COL$.

**Note 1.5** In the Graph Theory literature there are (at least) two kinds of coloring. We present them in this note so that if you happen to read the literature and they are using coloring in a different way then in these notes, you will not panic.

- **Vertex Coloring.** Usually one says that the vertices of a graph are $c$-colorable if there is a way to assign each vertex a color, using no more than $c$ colors, such that no two adjacent vertices (vertices connected by an edge) are the same color. Theorems are often of the form ‘if a graph $G$ has property BLAH BLAH then $G$ is $c$-colorable’ where they mean vertex $c$-colorable. We will not be considering these kinds of colorings.

- **Edge Colorings.** Usually this is used in the context of Ramsey Theory and Ramsey-type theorems. Theorems begin with ‘for all $c$-coloring of a graph $G$ BLAH BLAH happens’ We will be considering these kinds of colorings.
Def 1.6 Let \( n \in \mathbb{N} \). The \textit{complete graph on} \( n \) \textit{vertices}, denoted \( K_n \), is the graph

\[
V = [n] \\
E = \{\{i, j\} \mid i, j \in [n]\}
\]

Example 1.7 Let \( G \) be the complete graph on 10 vertices. Recall that the vertices are \( \{1, \ldots, 10\} \). We give a 3-coloring of the edges of \( G \):

\[
COL(\{x, y\}) =
\begin{cases} 
1 & \text{if } x + y \equiv 1 \pmod{3}; \\
2 & \text{if } x + y \equiv 2 \pmod{3}; \\
3 & \text{if } x + y \equiv 0 \pmod{3}.
\end{cases}
\]

Let's go back to our party! We can think of the 6 people as vertices of \( K_6 \). We can color edge \( \{i, j\} \) RED if \( i \) and \( j \) know each other, and BLUE if they do not.

Def 1.8 Let \( G = (V, E) \) be a graph, and let \( COL \) be a coloring of the edges of \( G \). A set of edges of \( G \) is \textit{monochromatic} if they are all the same color.

Let \( n \geq 2 \). Then \( G \) \textit{has a monochromatic} \( K_n \) if there is a set \( V' \) of \( n \) vertices (in \( V \)) such that

\begin{itemize}
  \item there is an edge between every pair of vertices in \( V' \):
    \( \{\{i, j\} \mid i, j \in V'\} \subseteq E \)
  \item all the edges between vertices in \( V' \) are the same color: there is some \( l \in [c] \) such that \( COL(\{i, j\}) = l \) for all \( i, j \in V' \). (I.e., \( V' \) is monochromatic.)
\end{itemize}

We now restate our 6-people-at-a-party theorem:

Theorem 1.9 \textit{Every 2-coloring of the edges of} \( K_6 \) \textit{has a monochromatic} \( K_3 \).

2 The Full Theorem

From the last section, we know the following:

\textit{If you want an} \( n \) \textit{such that you get a monochromatic} \( K_3 \) \textit{no matter how you 2-color} \( K_n \), \textit{then} \( n = 6 \) \textit{will suffice}.
What if you want to guarantee that there is a monochromatic $K_4$? What if you want to use 17 colors?

The following is known as Ramsey’s Theorem. It was first proved in [3] (see also [1], [2]).

**For all $c, m \geq 2$, there exists $n \geq m$ such that every $c$-coloring of $K_n$ has a monochromatic $K_m$.**

We will provide several proofs of this theorem for the $c = 2$ case. We will assume the colors are RED and BLUE (rather than the numbers 1 and 2). The general-$c$ case (where $c$ can be any integer $i \geq 2$) and other generalizations may show up on homework assignments.

## 3 Proof of Ramsey’s Theorem

**Theorem 3.1** *For every $m \geq 2$, $R(m)$ exists and $R(m) \leq 2^{2m-2}$.***

**Proof:**

Let $COL$ be a 2-coloring of $K_{2^{2m-2}}$. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{2^{m-1}},$$

and a sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots, V_{2^{m-1}},$$

that are based on $COL$.

Here is the intuition: Vertex $x_1 = 1$ has $2^{2m-2} - 1$ edges coming out of it. Some are RED, and some are BLUE. Hence there are at least $2^{2m-3}$ RED edges coming out of $x_1$, or there are at least $2^{2m-3}$ BLUE edges coming out of $x_1$. To see this, suppose, by way of contradiction, that it is false, and let N.E. be the total number of edges coming out of $x_1$. Then

$$\text{N.E.} \leq (2^{2m-3} - 1) + (2^{2m-3} - 1) = (2 \cdot 2^{2m-3}) - 2 = 2^{2m-2} - 2 < 2^{2m-2} - 1$$

Let $c_1$ be a color such that $x_1$ has at least $2^{2m-3}$ edges coming out of it that are colored $c_1$. Let $V_1$ be the set of vertices $v$ such that $COL(\{v, x_1\}) = c_1$. Then keep iterating this process.

We now describe it formally.
\(V_0 = [2^{2m-2}]\)
\(x_1 = 1\)

\(c_1 = \begin{cases} 
\text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \geq 2^{2m-3} \\
\text{BLUE} & \text{otherwise}
\end{cases}\)

\(V_1 = \{v \in V_0 \mid COL(\{v, x_1\}) = c_1\}\) (note that \(|V_1| \geq 2^{2m-3}\))

Let \(i \geq 2\), and assume that \(V_{i-1}\) is defined. We define \(x_i, c_i, \text{ and } V_i:\)

\(x_i = \) the least number in \(V_{i-1}\)

\(c_i = \begin{cases} 
\text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_i\}) = \text{RED}\}| \geq 2^{(2m-2)-i}; \\
\text{BLUE} & \text{otherwise.}
\end{cases}\)

\(V_i = \{v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i\}\) (note that \(|V_i| \geq 2^{(2m-2)-i}\))

How long can this sequence go on for? Well, \(x_i\) can be defined if \(V_{i-1}\) is nonempty. Note that

\(|V_{2m-2}| \geq 2^{(2m-2)-(2m-2)} = 2^0 = 1\)

Thus if \(i-1 = 2m-2\) (equivalently, \(i = 2m-1\)), then \(V_{i-1} = V_{2m-2} \neq \emptyset\), but there is no guarantee that \(V_i (= V_{2m-1})\) is nonempty. Hence we can define

\[x_1, \ldots, x_{2m-1}\]

Consider the colors

\(c_1, c_2, \ldots, c_{2m-2}\)

Each of these is either RED or BLUE. Hence there must be at least \(m-1\) of them that are the same color. Let \(i_1, \ldots, i_{m-1}\) be such that \(i_1 < \cdots < i_{m-1}\) and

\(c_{i_1} = c_{i_2} = \cdots = c_{i_{m-1}}\)

Denote this color by \(c\), and consider the \(m\) vertices

\[x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_{m-1}+1}\]
To see why we have listed \( m \) vertices but only \( m - 1 \) colors, picture the following scenario: You are building a fence row, and you want (say) 7 sections of fence. To do that, you need 8 fence posts to hold it up. Now think of the fence posts as vertices, and the sections of fence as edges between successive vertices, and recall that every edge has a color associated with it.

**Claim:** The \( m \) vertices listed above form a monochromatic \( K_m \).

**Proof of Claim:**

First, consider vertex \( x_{i_1} \). The vertices

\[x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_{m-1}+1}\]

are elements of \( V_{i_1} \), hence the edges

\[\{x_{i_1}, x_{i_2}\}, \ldots, \{x_{i_1}, x_{i_{m-1}}\}, \{x_{i_1}, x_{i_{m-1}+1}\}\]

are colored with \( c_{i_1} \) (\( = c \)).

Then consider each of the remaining vertices in turn, starting with vertex \( x_{i_2} \). For example, the vertices

\[x_{i_3}, \ldots, x_{i_{m-1}}, x_{i_{m-1}+1}\]

are elements of \( V_{i_2} \), hence the edges

\[\{x_{i_2}, x_{i_3}\}, \ldots, \{x_{i_2}, x_{i_{m-1}}\}, \{x_{i_2}, x_{i_{m-1}+1}\}\]

are colored with \( c_{i_2} \) (\( = c \)).

*End of Proof of Claim.*

### 4 Proof of the Infinite Ramsey Theorem

We now consider infinite graphs.

**Notation 4.1** \( K_N \) is the graph \( (V, E) \) where

\[V = \mathbb{N} \]

\[E = \{\{x, y\} \mid x, y \in \mathbb{N}\}\]
Def 4.2 Let $G = (V, E)$ be a graph with $V = \mathbb{N}$, and let $\text{COL}$ be a coloring of the edges of $G$. A set of edges of $G$ is monochromatic if they are all the same color (this is the same as for a finite graph).

$G$ has a monochromatic $K\mathbb{N}$ if there is an infinite set $V'$ of vertices (in $V$) such that

- there is an edge between every pair of vertices in $V'$
- all the edges between vertices in $V'$ are the same color

Theorem 4.3 Every 2-coloring of the edges of $K\mathbb{N}$ has a monochromatic $K\mathbb{N}$.

Proof:
(Note: this proof is similar to the proof of Theorem 3.1.)

Let $\text{COL}$ be a 2-coloring of $K\mathbb{N}$. We define an infinite sequence of vertices, $x_1, x_2, \ldots,$

and an infinite sequence of sets of vertices, $V_0, V_1, V_2, \ldots,$

that are based on $\text{COL}$.

Here is the intuition: Vertex $x_1 = 1$ has an infinite number of edges coming out of it. Some are RED, and some are BLUE. Hence there are an infinite number of RED edges coming out of $x_1$, or there are an infinite number of BLUE edges coming out of $x_1$ (or both). Let $c_1$ be a color such that $x_1$ has an infinite number of edges coming out of it that are colored $c_1$. Let $V_1$ be the set of vertices $v$ such that $\text{COL}(\{v, x_1\}) = c_1$. Then keep iterating this process.

We now describe it formally.

$$V_0 = \mathbb{N}$$
$$x_1 = 1$$

$$c_1 = \begin{cases} 
\text{RED} & \text{if } |\{v \in V_0 \mid \text{COL}(\{v, x_1\}) = \text{RED}\}| \text{ is infinite} \\
\text{BLUE} & \text{otherwise}
\end{cases}$$

$$V_1 = \{v \in V_0 \mid \text{COL}(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \text{ is infinite)}$$
Let $i \geq 2$, and assume that $V_{i-1}$ is defined. We define $x_i$, $c_i$, and $V_i$:

$$x_i = \text{the least number in } V_{i-1}$$

$$c_i = \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid \text{COL}(\{v, x_i\}) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$V_i = \{v \in V_{i-1} \mid \text{COL}(\{v, x_i\}) = c_i\} \quad \text{(note that } |V_i| \text{ is infinite)}$$

How long can this sequence go on for? Well, $x_i$ can be defined if $V_{i-1}$ is nonempty. We can show by induction that, for every $i$, $V_i$ is infinite. Hence the sequence

$$x_1, x_2, \ldots,$$

is infinite.

Consider the infinite sequence

$$c_1, c_2, \ldots$$

Each of the colors in this sequence is either RED or BLUE. Hence there must be an infinite sequence $i_1, i_2, \ldots$ such that $i_1 < i_2 < \cdots$ and

$$c_{i_1} = c_{i_2} = \cdots$$

Denote this color by $c$, and consider the vertices

$$x_{i_1}, x_{i_2}, \cdots$$

Using an argument similar to the one we used in the proof of Theorem 3.1 (to show that we had a monochromatic $K_m$), we can show that these vertices form a monochromatic $K_N$.

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**References**

