The Hamiltonian Cycle Problem is NP-Complete

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CMSC 452

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Section 1

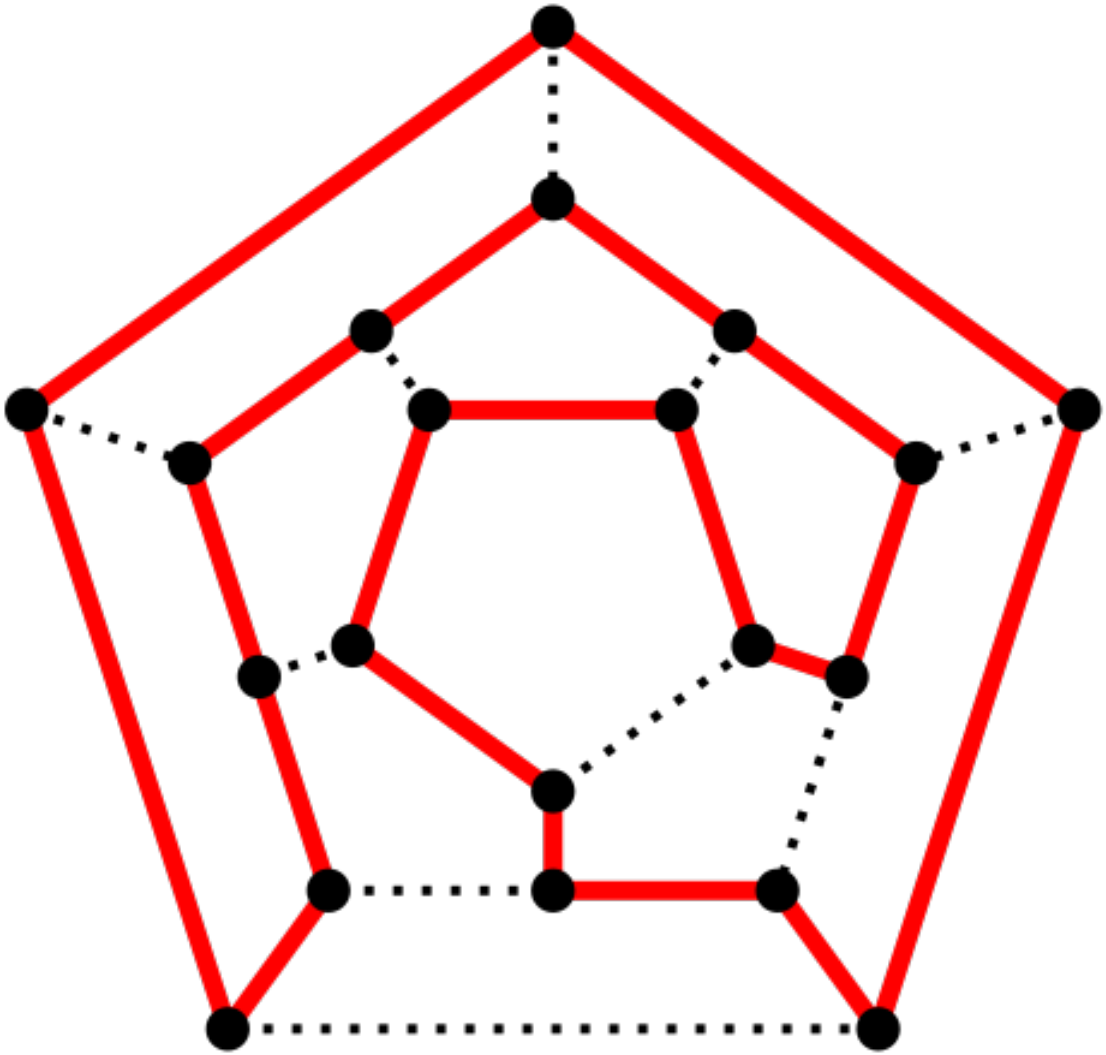
Outline

Section 2

Introduction

What is a Hamiltonian Cycle

A cycle through a graph $G = (V, E)$ that touches every vertex once.



Hamiltonian Path $\in NP$

1. The certificate: a path represented by an ordering of the verticies

Hamiltonian Path $\in NP$

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2. Verify:
   - Each node is in the path once
   - An edge exists between each consecutive pair of nodes

Section 3

3-SAT $\leq_P$ Directed Ham Path

Subsection 1

Procedure

Procedure

1. Start with a 3-CNF formula

$$\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \cdots \wedge (a_k \vee b_k \vee c_k)$$

Procedure

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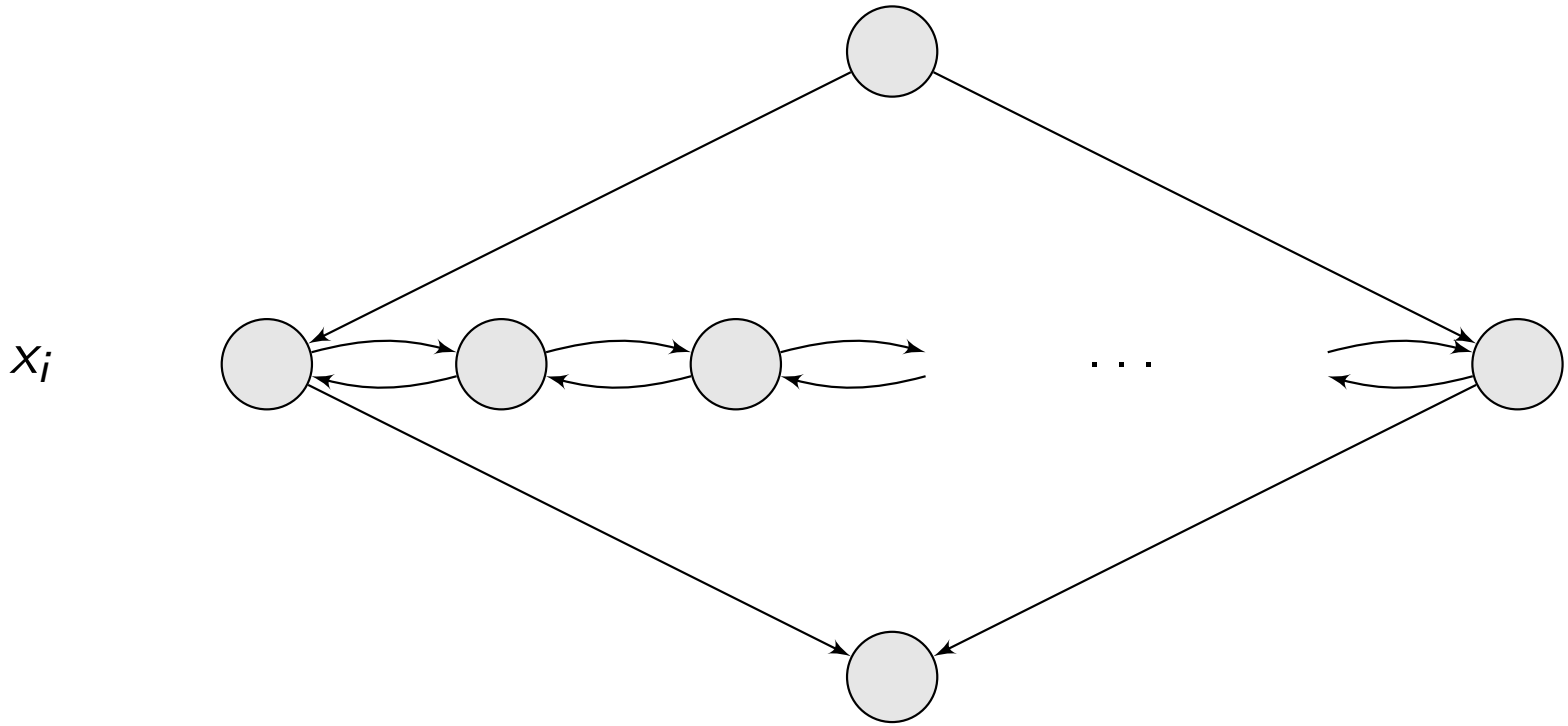
2. Create a graph G that has a Hamiltonian Path iff ϕ is satisfiable

Subsection 2

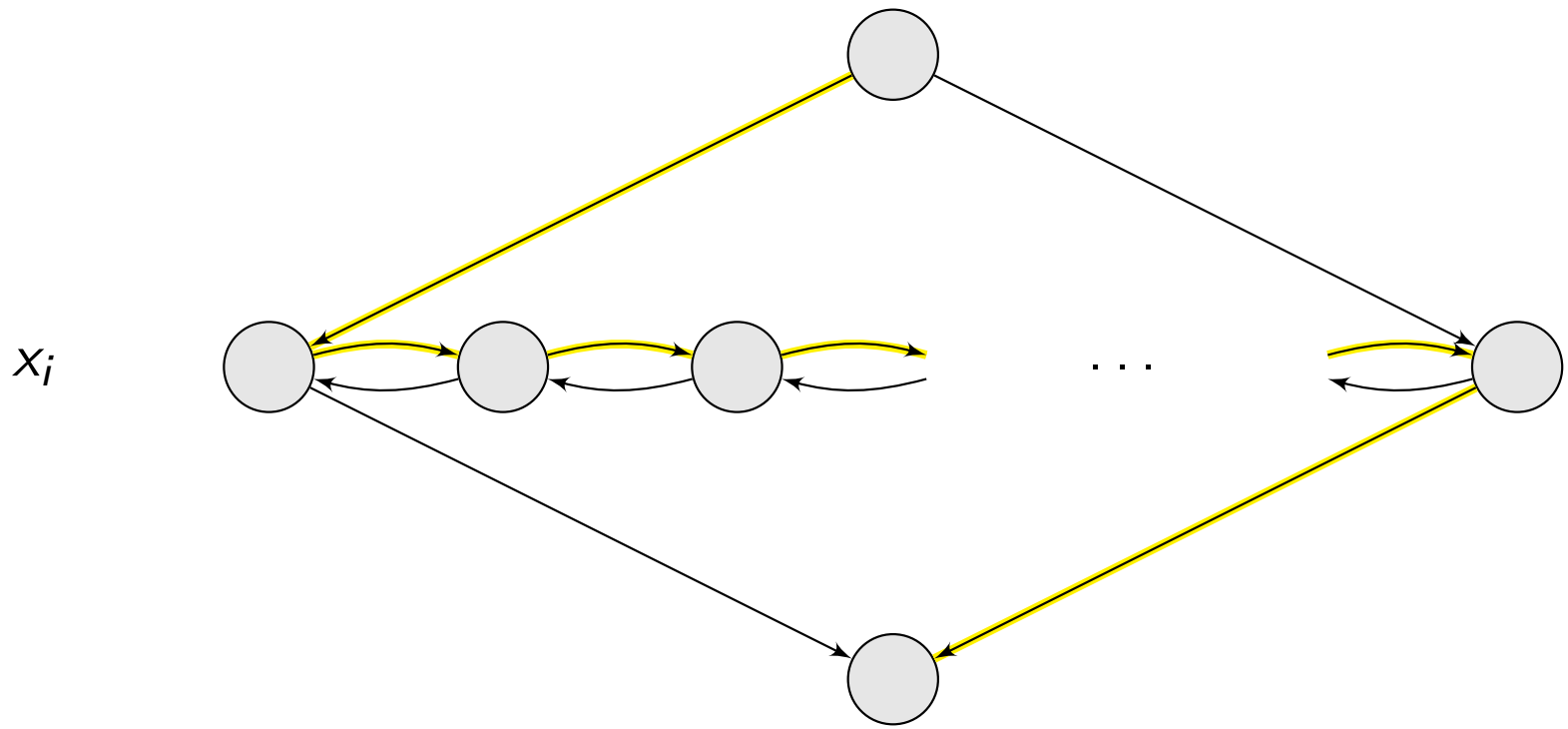
Construction

Variables

Represent each variable x_i with a gadget with $3k + 3$ nodes:



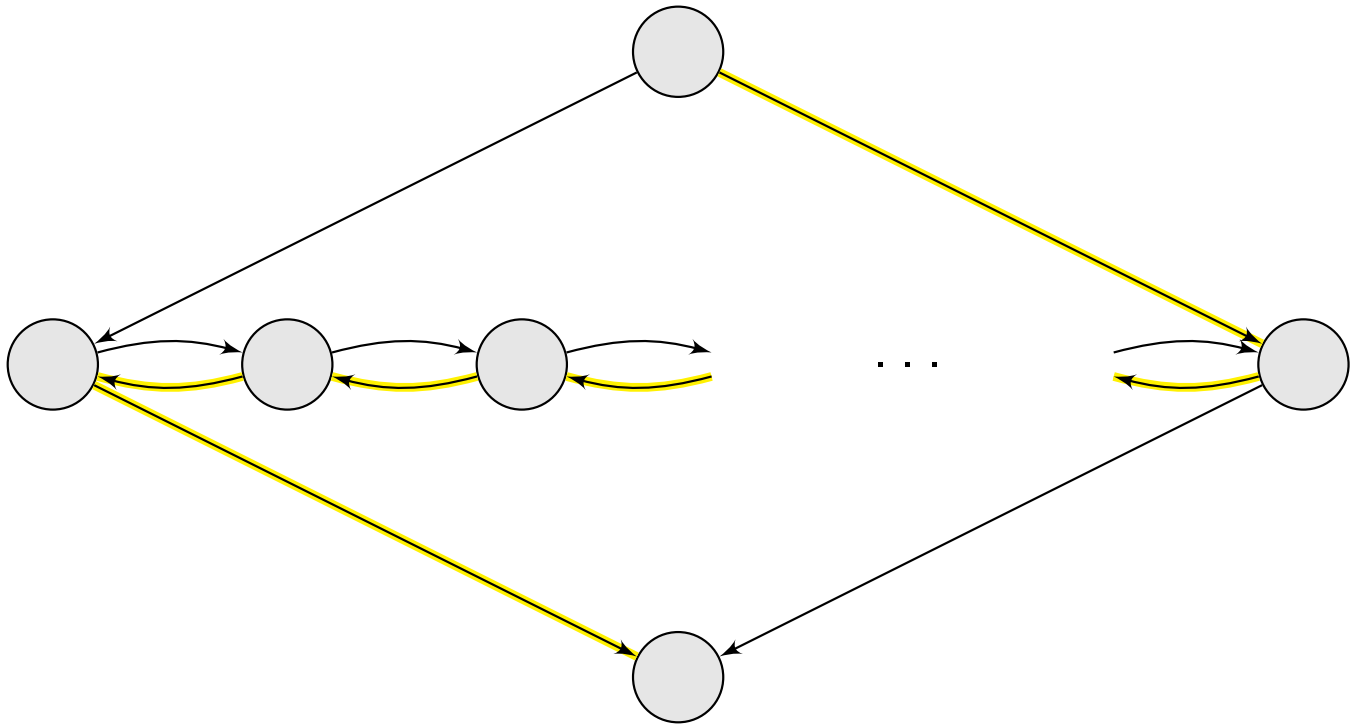
2 Ways



True:

2 Ways

x_i

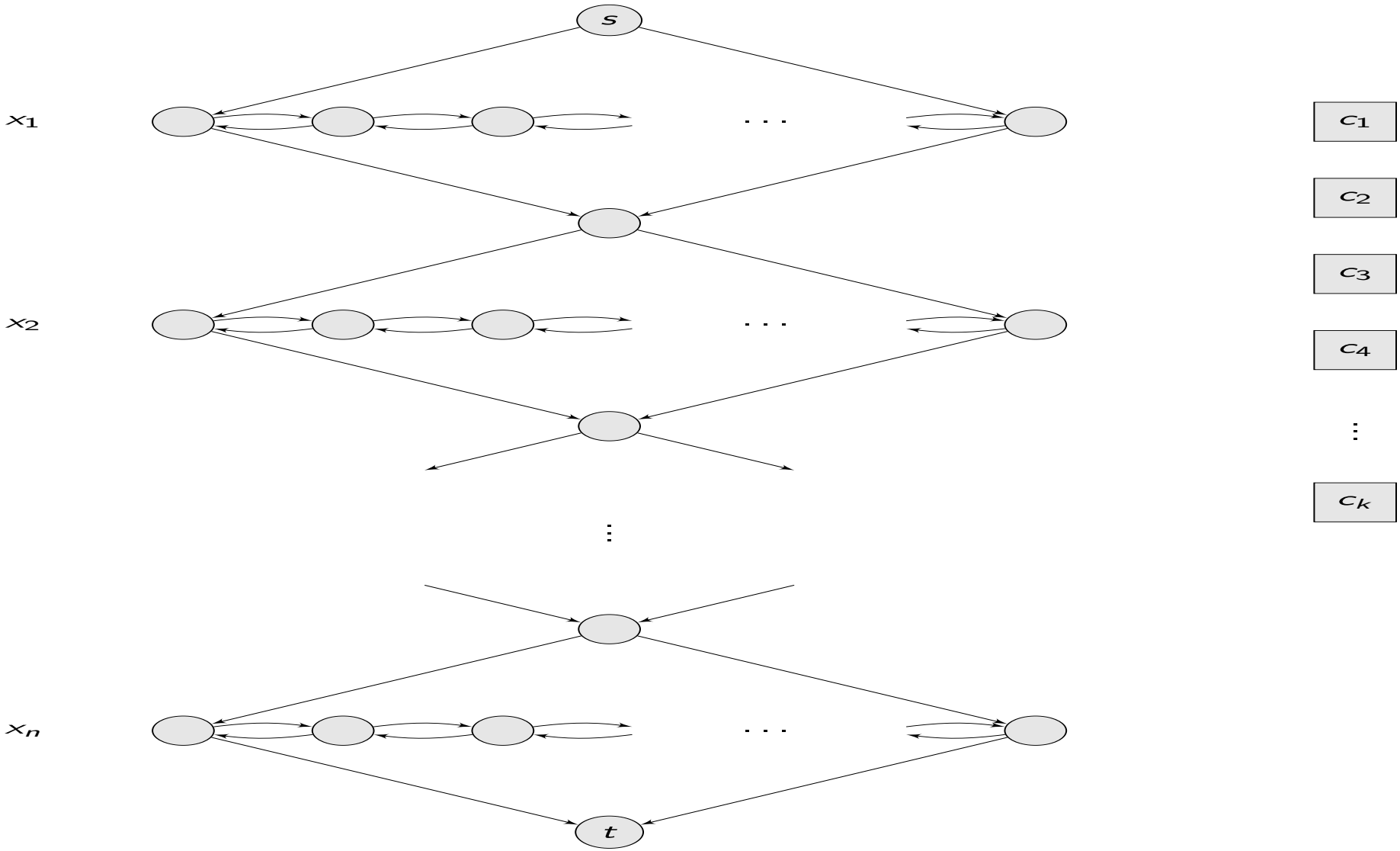


False:

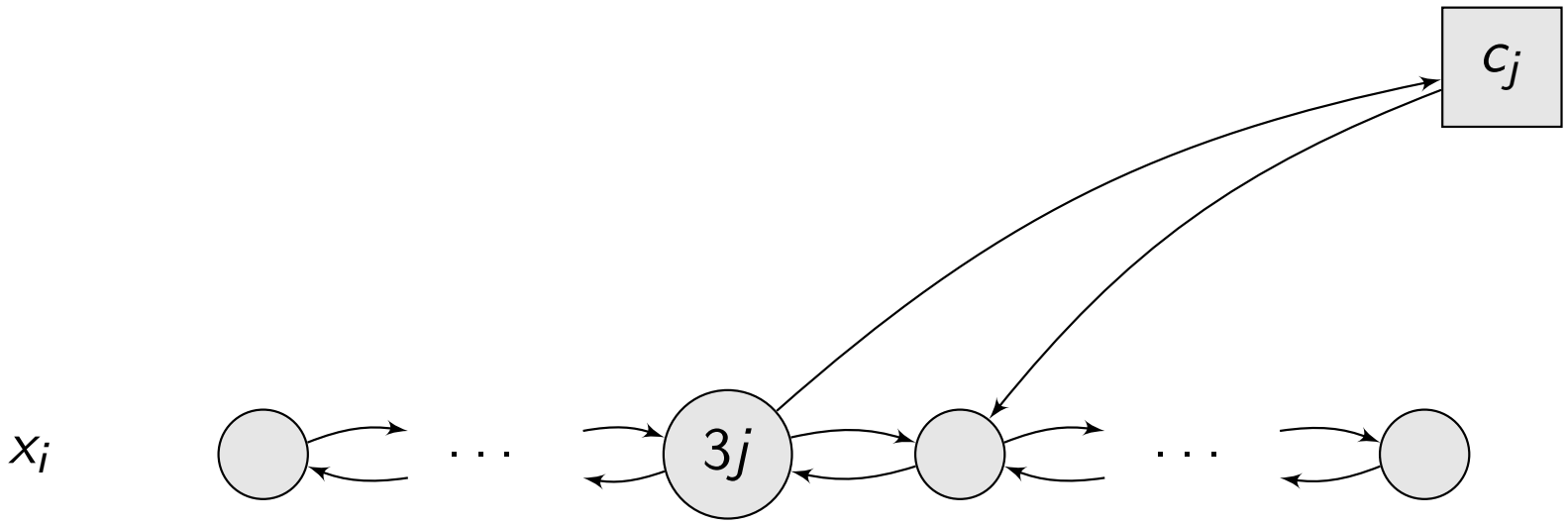
Clauses

Each clause c_j is represented as a single node:

Connect the Variables



Connect the Clauses



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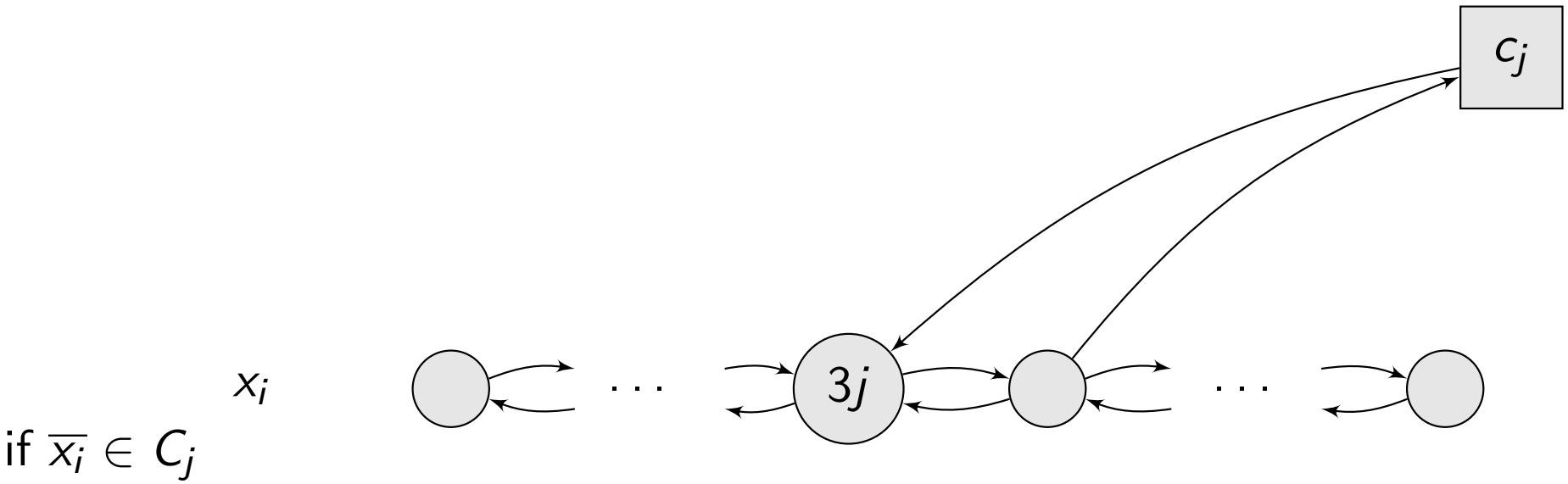
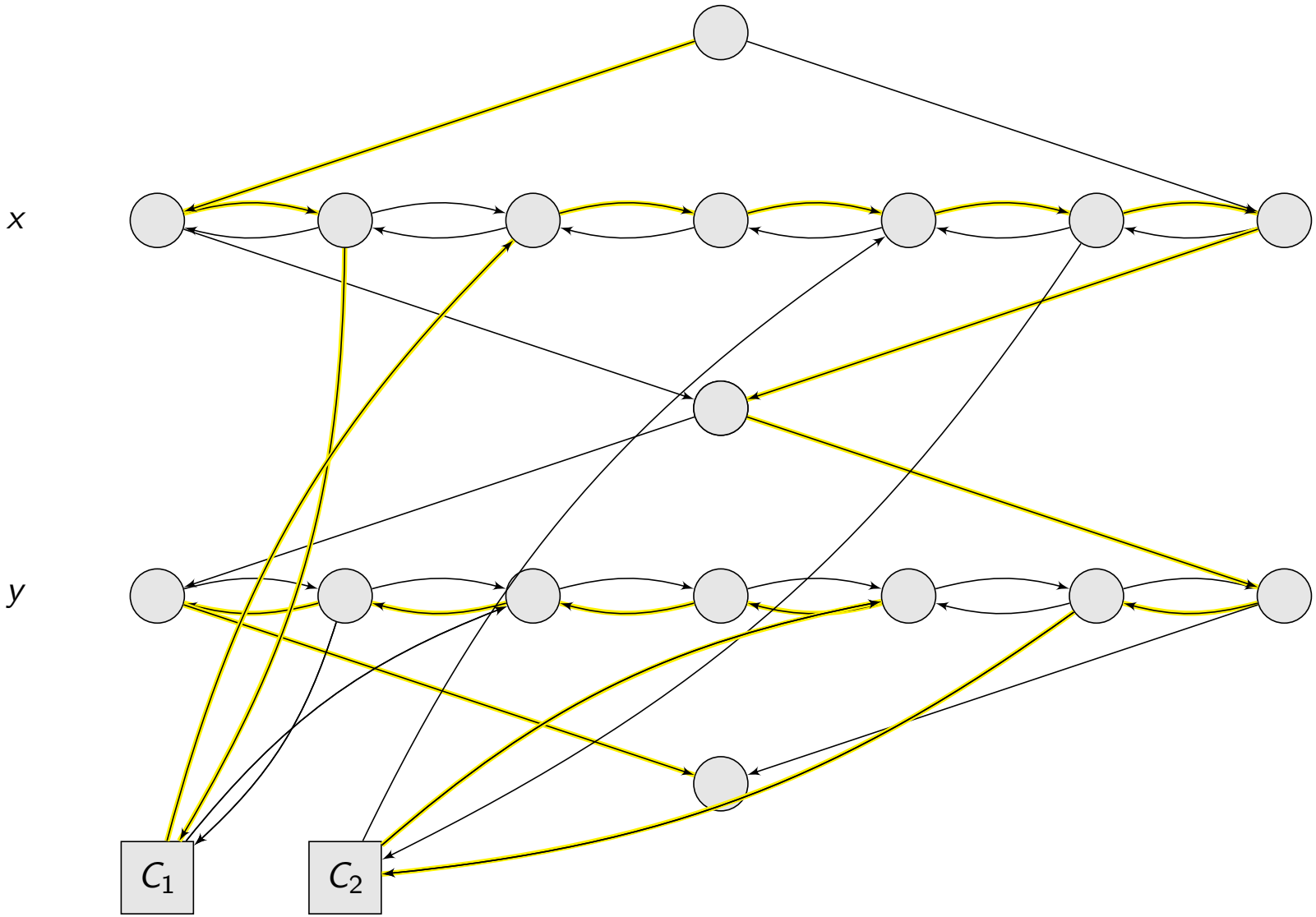


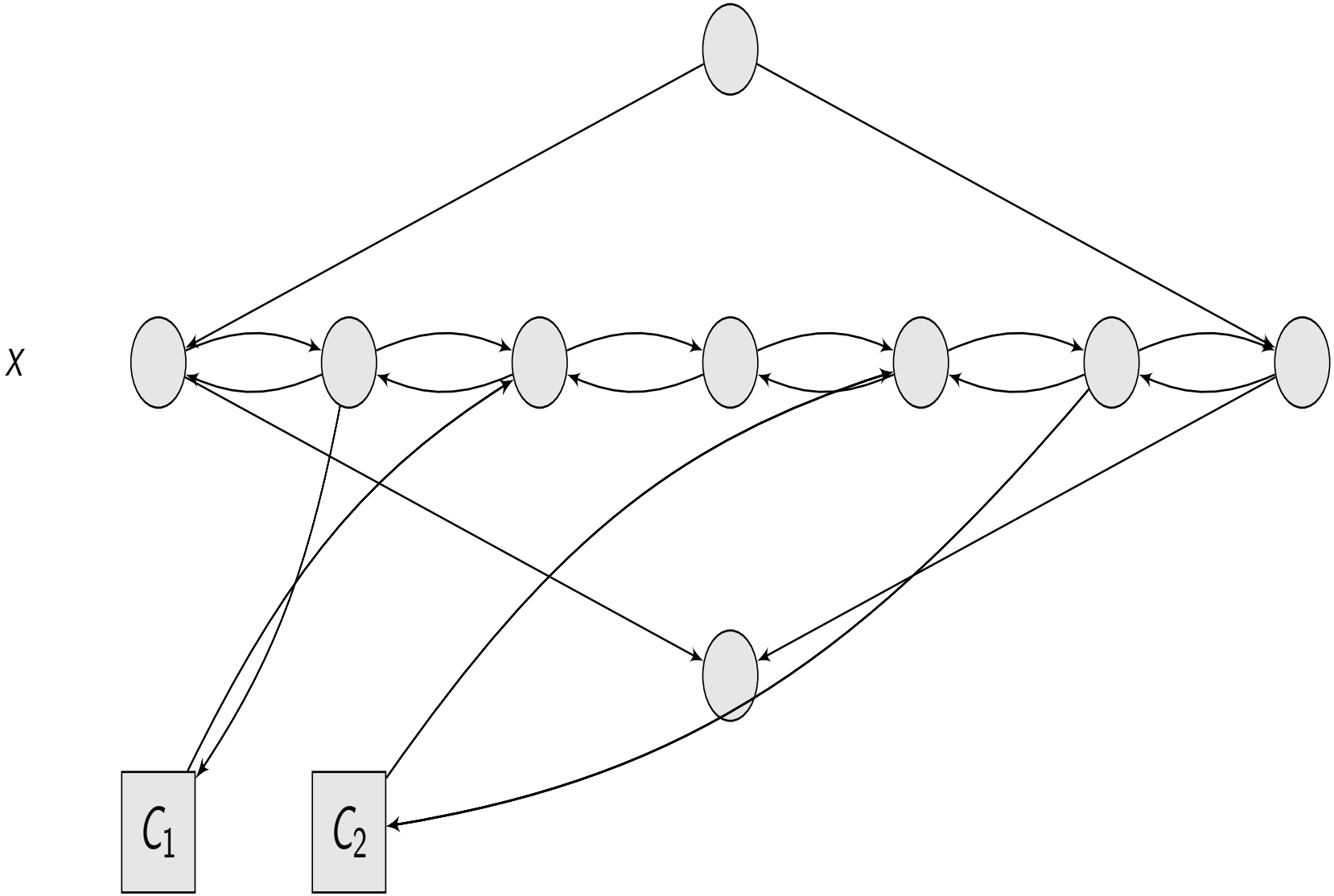
Subsection 3

Examples

$(x \vee y) \wedge (\overline{x} \vee \overline{y})$



$x \wedge \overline{x}$



Subsection 4

A Dialog

At this point...

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- Scott: Maybe in Prolog.

At this point...

- Gasarch [to class]: So, could you code this up?
- Class: Yes!
- Gasarch: Would you want to?
- Class: No...
- Scott: Maybe in Prolog.
- Liz: If you paid me.

At this point...

- Gasarch [to class]: So, could you code this up?
- Class: Yes!
- Gasarch: Would you want to?
- Class: No...
- Scott: Maybe in Prolog.
- Liz: If you paid me.
- Gasarch: What happened to the love of computer science! When Clyde was an undergrad... The answer is 0 or on the board!... blah blah blah

Section 4

Hamiltonian Path $\leq_P$ Hamiltonian Cycle

...

Question 4 on the Final

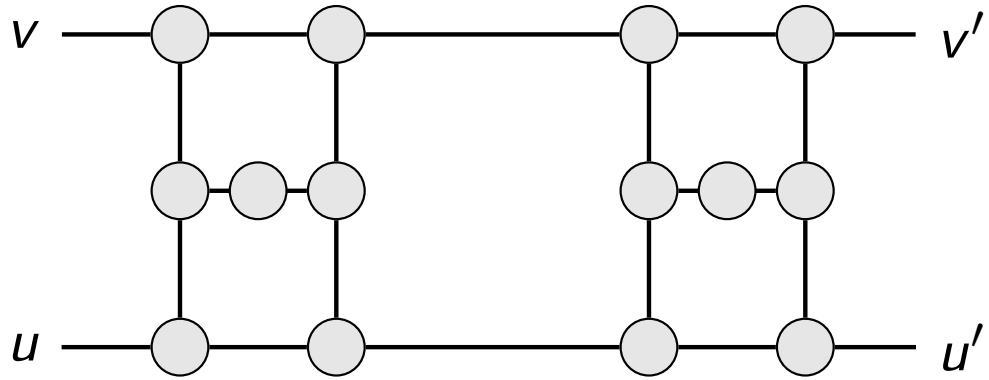
Section 5

3-SAT $\leq_P$ Undirected Planar Hamiltonian Cycle

Subsection 1

Gadgets

Or



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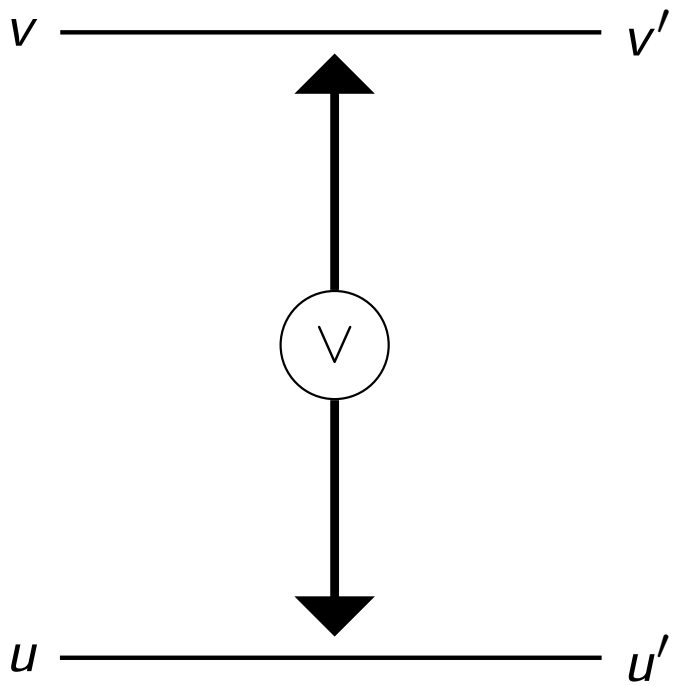
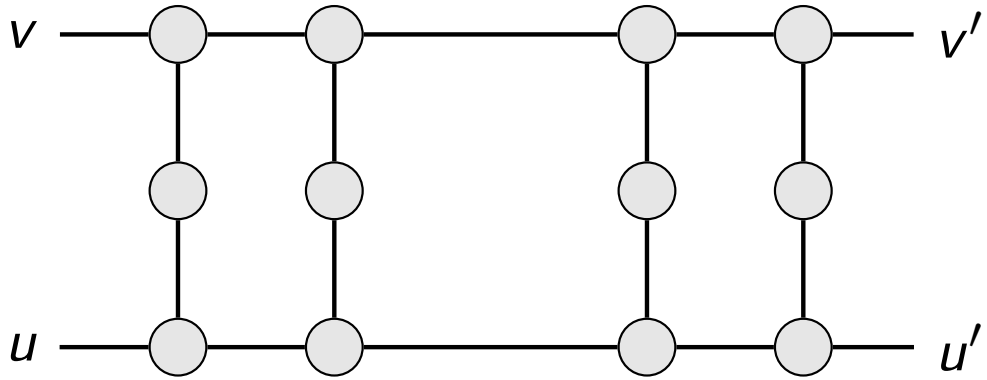
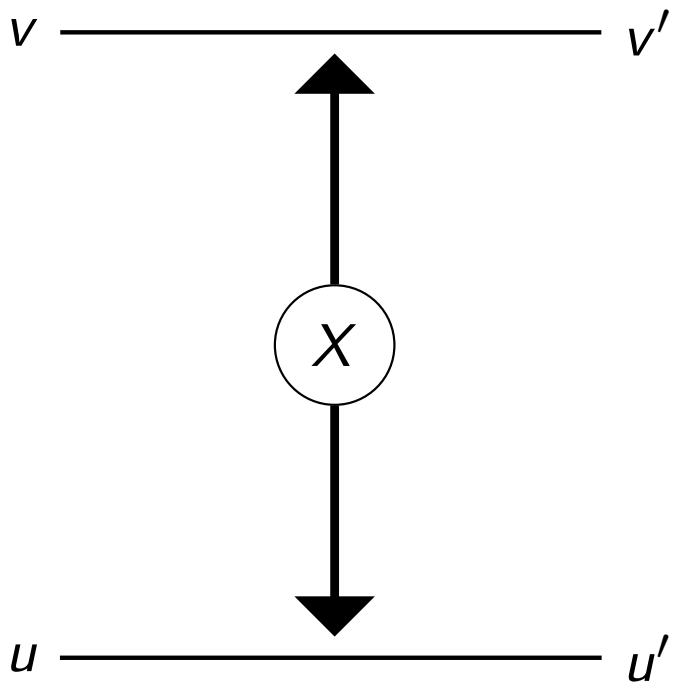


X-Or



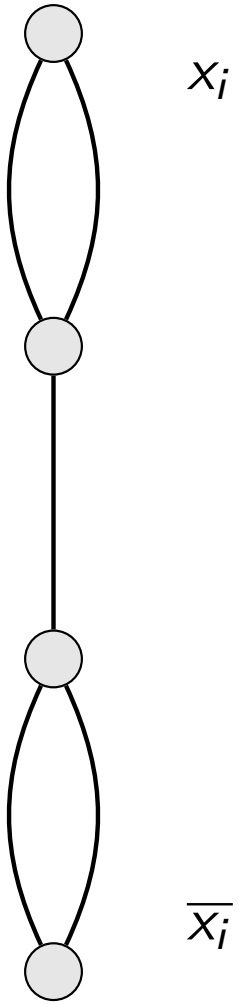
X-Or



Subsection 2

Construction

Variables



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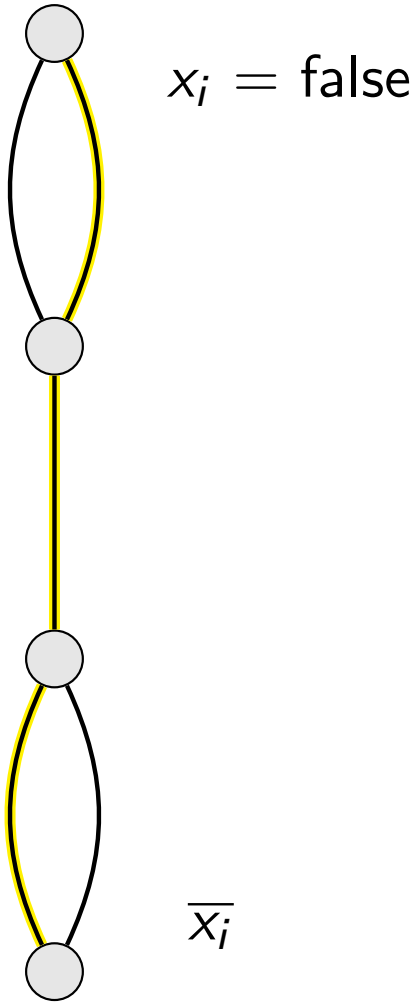

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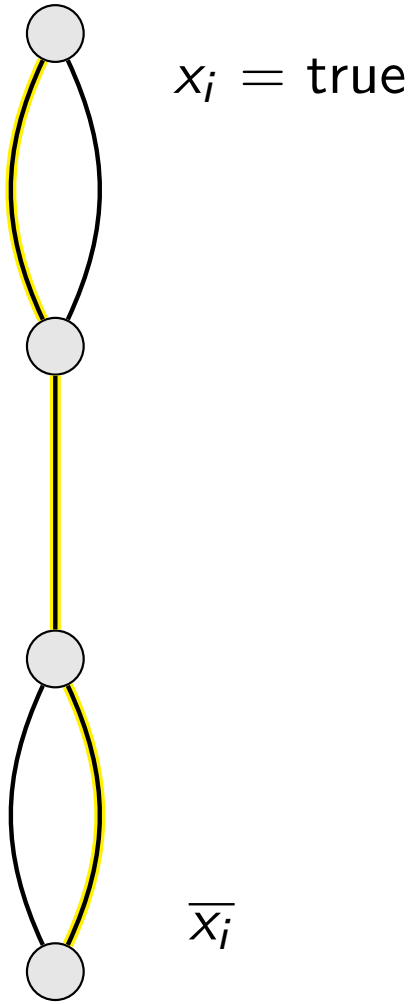

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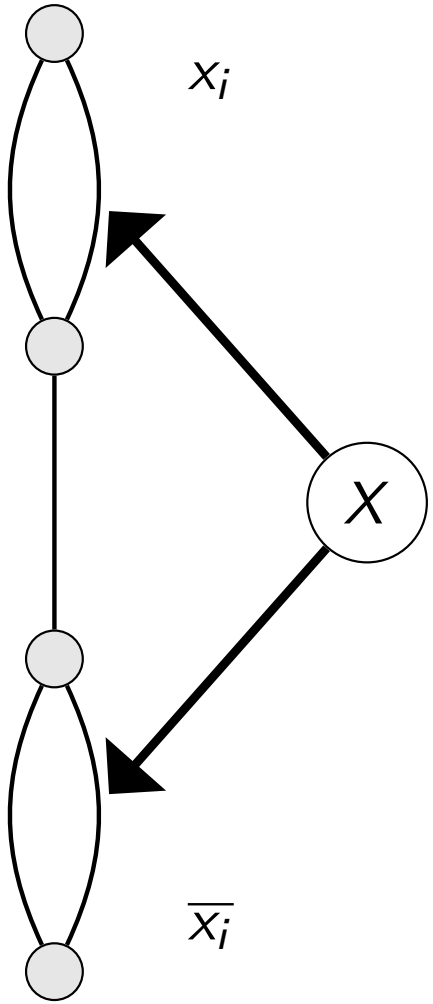
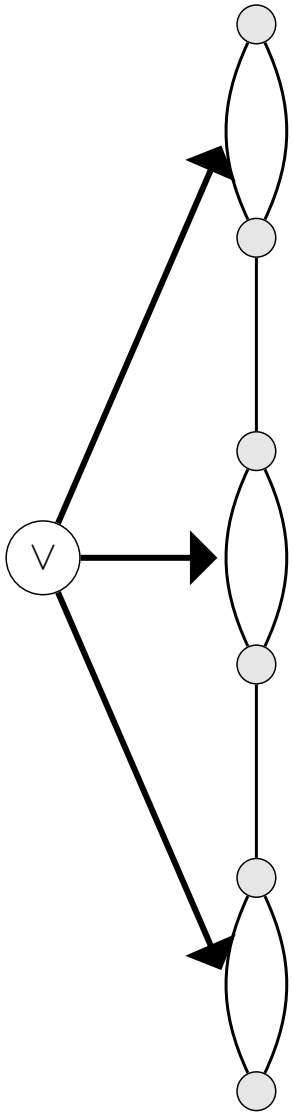
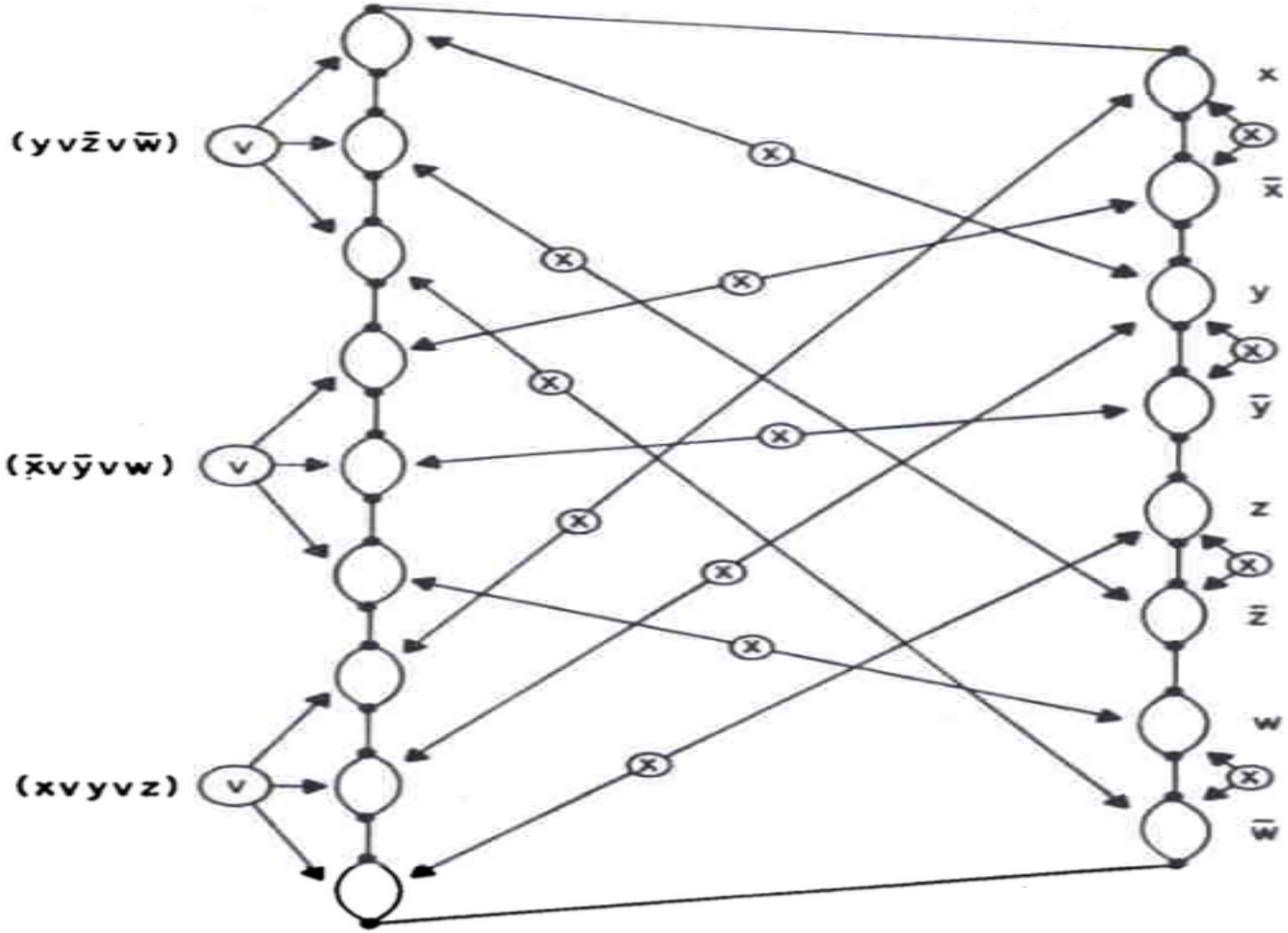


Clauses



$$(x \vee y \vee z) \wedge (\overline{x} \vee \overline{y} \vee w) \wedge (y \vee \overline{z} \vee \overline{w})$$



That's BS Man!