Joint Review[1] of
The Honor Class: Hilbert's Problems and Their Solver
Author: Ben Yandell
Publisher: A.K Peters, 2002, $25.00, 486 pages

and
Mathematical Developments arising from Hilbert's Problems
Edited by Felix Browder
Publisher: American Math Society, 1974, $40.00
628 pages in Two Volumes
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1 Overview

In the year 1900 Hilbert posed 23 problems for mathematicians to work on over the next 100 years. These problems have shaped mathematics and (to some extent) guided what mathematicians work on. To solve one is, to paraphrase Hermann Weyl, to enter *The Honor Class*.

In this review I first say a little about each of the two books and then do a joint summary by saying what each one said about some of the problems. I chose to talk about the problems that I think will interest my readers.

1.1 The Honor Class

The book *The Honor Class* is about the people who solved the problems and the times they lived in. There is some mathematics in it as well—enough so that readers of this column can understand the problems and why they were important. A layperson can also get something out of the mathematics, though that varies from problem to problem. This book was written in 2002; however, for the most part, it is not dated.

Why do we study history? Some read history to get a better idea of how we got to where we are now. When we see the problems people worked on, and why, we are informed as to (1) why we are working on similar or different problems, (2) why we are or are not using certain approaches, and (3) what our expectations are and perhaps should be.

I heard a historian say that reading history is like going to a different country— when you go back to your own country you see things in a different light since you've seen an alternative way of doing things. This applies to history of math in two ways: (1) the math itself— seeing how they thought of math and how we do is enlightening. (2) how the culture of research has changed— Jews and Women were often banned from universities; people communicated by postal mail (you can ask your grandparents what that is); people would argue furiously about the degree of rigor needed in a proofs; and nonconstructive proofs were considered controversial. We do indeed live in different times. I will go out on a limb and say MUCH better times.

The Honor Class gives context for the math presented and presents how math was done in those days. There is also a good deal of history in this book. Not history of the sort *The Hitler-Stalin* *pact was signed in April 23, 1939 and had drastic consequences,* but history of the sort that tells what ordinary citizens (largely mathematicians) were doing in those times. World War I, World War II, and the Cold war permeate the entire book. This book *is not* about the moral dilemmas of what a mathematician (or anybody) should do when working in (say) Nazi Germany (for that see *Mathematics under the Nazis* by Sanford Segal). This book *is* about the particular people involved with Hilbert's Problems- their math and their lives. The moral questions arise naturally.

The Honor Class is organized into eight sections, each of which have chapters. The book is *not* in order of the problems. Instead it groups the problems into natural categories such as *The Foundation Problems.*

1.2 Mathematical Developments arising from Hilbert's Problems

The book *Mathematical Developments arising from Hilbert's Problems* is the proceedings of a 1974 conference on Hilbert's Problems. There is one article about each problem except (1) there is no article on Problem 3 (showing that the volume formula for triangular pyramids requires continuous methods), and (2) there are two articles on Problem 8 (Riemann Hypothesis). The articles are on the problems and what happened *after* they were solved (if they were) or what mathematics has been created to *try to solve it* if it hasn't. The former goal is very interesting because it reminds us that once a problem is solved it is not the end of the story. This book was written in 1974; however, while I am sure some of it dated, much of it is not. There are two reasons for this: (1) Math moves rather slowly, and (2) the material about the problems, how they were solved, what happens next is still interesting.

2 Summary of Both Books

I abbreviate *Hilbert's nth Problem* by Hn, *The Honor Class* by *Honor*, and *Mathematical Developments arising from Hilbert's Problems* by *Math Dev.*

H1: The Continuum Hypothesis:

The Continuum Hypothesis (CH) is the statement that every infinite subset of the reals is of cardinality either that of N or that of R. Hilbert wanted a proof of CH though one suspects he would have been happy (though surprised) with a proof of $\neg CH$. He certainly thought it was either true or false. However, CH was shown independent of Set Theory by Godel and Cohen. This would have shocked Hilbert.

In *Honor* the chapter on Cantor reveals just what an advance Cantor made. In the 1800's mathematicians were grappling with the basic definitions of sets and functions. At one time only functions that were differentiable were considered functions. It is hard for us, 21st century people, to grasp that *function* meant *function that comes up in nature.* Dirichlet is often credited with the modern definition of function (though this is not clear). Cantor's set theory allowed virtually ANY collection of objects to be a set. This was revolutionary. Many mathematicians objected since they only wanted *well behaved sets* to be considered. Poincare famously said *Set theory is a disease from which mathematics will one day recover.* Cantor had a hard time forming a counterargument since he had various mental problems. Even so, Cantor's viewpoint won out and is now accepted today without controversy.

What did the Catholic Church think of Cantor's work? If they opposed it I am sure we would all know that. However, Cantor contacted them and they decided that it was good philosophy and compatible with the Church's position. Not as interesting a story as Galileo (allegedly) saying under his breath "and yet it moves" but worth knowing.

The chapter of *Honor* on H1 discusses Godel's unhappy life. He was born in Germany and thought of himself as German; however, he had Jewish friends and looked Jewish; hence he was forced to leave the country. This was difficult as there was a blockade at the time, though he managed to get to Princeton. He also had mental problems which may have lead to his death (at 72 years old) of starvation.

In *Math Dev* Donald Martin has an article that talks about how CH may still be resolved some day via large cardinal axioms or some version of the Axiom of Determinacy (Donald Martin had recently proven AD for Borel Sets). AD does imply the version of CH stated here though this is sometimes called the weak version (the strong version is that the first uncountable ordinal is the same size as R). The article seems dated in a charming way since (I think) most set theorists nowadays think of CH as not having an answer.

For a modern take on why set theorists believe the axioms they do, see Penelope Maddy's articles *Believing the Axioms I, II* in *Journal of Symbolic Logic* Vol 53, No 2, 481–511, 1988, Vol 53, No 3, 736–764, 1988. Or go to the her webpage to find a free copy that is better since there are corrections.

H2: Prove that Arithmetic is Consistent. This was shown to be impossible by Godel.

The chapter in *Honor* on Godel also talks about Hilbert and the Second problem before getting to Godel. We see that an objection to Hilbert's axiomatic approach was that Hilbert wanted (or perhaps this was a caricature) all axiom systems to be considered equally worth working in. We DO have some of this attitude now with people asking *can I prove this without the Axiom of Choice?* rather than saying *is this TRUE.*

In *Math Dev* Georg Kreisel discusses how close one can come to realizing Hilbert's dream. In particular he discusses which mathematical theories are decidable, which ones are not, and how to tell the difference. He looks at real closed fields, certain geometries that are decidable, and Presburger arithmetic. The article also points out that what Hilbert asked for needs to be better refined to be understood.

H3: Show that the following Theorem requires continuous methods: *Two triangular pyramids of equal height and base have the same volume.* Proven by Max Dehn.

If you go by Google Hits this is the least well known of Hilbert's problems. This may be because it was solved very shortly before it was posed. (Recall that communication was slower in those days.) There is no chapter on it in *Math Dev.*

Hilbert (and most people) liked to have theorems in field X proved using tools just from field X. The theorem stated in my description of H3 was an example of this issue: a theorem in solid geometry that uses calculus to prove it. In Hilbert's day it was thought that such a proof was needed. And they were right.

Honor has a chapter about Max Dehn's fascinating life. By all accounts he was brilliant. He was a Jew in Germany during the Nazi era and had to flee. He managed to get out and teach at *Black Mountain College* which was a very odd place in that it ran by consensus of the faculty and the students. It no longer exists.

When one reads accounts of people who fled from Nazi Germany one tends to only read of those that escaped. Many did not. As a reminder of this *Honor* notes that one of Dehn's colleagues, Paul Epstein, also Jewish, killed himself rather than be taken by the Gestapo.

H4: Explore Different Geometries that use Different Notions of Distance.

This problem is more of a research plan than a problem. The Wikipedia entry on H4 said that the problem is too vague to be ever be considered solved. Pogorelov wrote a book in 1979 on the problem which formulates it in rigorous terms and solves it, using the work of Busemann. *Honor* credits the solution to *Busemann, Pogorelov,* ... (I do not know what the ... means.)

In *Math Dev* Busemann writes about how the problem was harder and less well defined than Hilbert intended. To quote: *The Fourth Problem concerns the Geometries in which ordinary lines, i.e., lines of an n-dimensional (real) projective space P^n or pieces of them are the shortest curves of geodesics. Specifically, Hilbert asks for the construction of all of these metrics and the study of the individual geometries. It is clear from Hilbert's comments that he was not aware of the immense number of these metrics, so that the second part of the problem is not a well posed question and has inevitably been replaced by the investigation of special, or special classes, of interesting geometries.* The article then goes on to rephrase what Hilbert should have asked and connect it to the calculus of variations.

In my first draft of this review I had a comment like *H4 began as a question on foundations but then lead to some real math.* However, this is a very 21st century view. The distinction between *foundations* and *real math* is not appropriate for that era and perhaps is sharper than it should be in ours.

H6: Axiomitize Physics Good luck.

This problem is more of a research plan than a problem. The chapter in *Honor* about this problem is not about people. Its about the on-again off-again marriage between Mathematics and Physics. Mathematics and Physics were joined at the hip before the 20th century in ways that are hard for us to imagine now. The article also claims that they are now reuniting. This may be; however, it will never be what it once was. As a major example or how closely they were connected, as noted in this review regarding H1, at one time *function* meant *function that occurs in nature.* As a minor example of how closely they were connected note that the Putnam exam used to have problems on Physics but now they are gone, replaced by problems on combinatorics [2]

Hilbert himself mentioned probability and the kinetic theory of gases as already having a rigorous treatment. Even though probability is not a branch of physics his point was that a previously non-rigorous study had been made rigorous. In *Math Dev* Wightman writes an article that has two parts. First he briefly surveys Hilbert's own work on gases, radiation, relativity, and quantum mechanics. Then he describes, at length, recent (circa 1974) attempts to axiomitize quantum mechanics and quantum field theory. This is the longest chapter by far in the book– around 90 pages. It is not a light read.

H7: Show that if a is algebraic and b is algebraic and irrational then a^b is transcendental. (For example show that $2^{\sqrt{2}}$ is transcendental.) The general theorem was shown by Gelfond, Schnider, and Siegel.

Siegel lead an interesting life. He was German. He avoided serving in WW I by pretending to be mentally ill. He fled Nazi Germany shortly before WW II but after the war went back. He truly wanted to revive Mathematics in Germany. He worked in number theory but also in mechanics. This type of well roundedness is rare today. He was also a member of the Frankfurt Historical

[2]See `http://blog.computationalcomplexity.org/2009/03/putnam-exam-some-thoughts.html` for a full study of this.

Math Society which read the original papers in the original languages and had an attitude of not publishing too much. (Dehn was also a member.)

Schneider was a student of Siegel's. He joined the Germany Army in WW II even though he disagreed with the Nazis. He hoped this would get him better treatment from them but it did not—his Habilitation[3] was rejected without being read. After the war his fortunes improved considerably.

Gelfond was Russian and had to deal with the oddities of the Soviet system. Decisions were capricious and arbitrary (e.g., political) as to who got promoted and who got jobs.

In *Math Dev* Tijdeman gives a survey of more recent work that extends Gelfond's technique. We give an example of different types of theorems.

1. Let $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers. If $\log(\alpha_1), \ldots, \log(\alpha_n)$ are linearly ind. over Q then they are linearly ind. over A (the algebraic numbers).

2. Let α, β be algebraic with $\alpha \neq 0, 1$. Assume that β has degree at least $2^n - 1$ with $n \geq 2$. At least n of the following numbers are algebraically independent: $\{\alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^N}\}$ where $N = 2^{n+1} - 4$.

3. Let $k \in \mathsf{Z}$. Every solution to $y^2 = x^3 + k$ satisfies $|x|, |y| \leq \exp((10^{10}|k|)^{10^4})$. This can be seen as solving a subcase of H10 in the positive direction.

Both books tell the following story: Hilbert in 1920 predicted that (1) he would live to see the Riemann hypothesis solved, (2) some younger people in the audience would live to see Fermat's last theorem proved, but (3) H7 would not be seen by anybody in the audience. In reality (1) H7 was solved in 1934 (in Hilbert's lifetime), (2) Fermat's last theorem was solved in 1994 (if there was a 20 year old in the audience then they would have to live to 94 to see that there was a proof), and (3) the Riemann hypothesis is still open.

H10: Give an algorithm that will, given a polynomial in $Z[x_1, \ldots, x_n]$, determines if it has an integer solution. This was meant to be a problem in Number Theory but it is undecidable so it became a problem in logic. While this may have surprised Hilbert note that, in Hilbert's address, he noted the following which I quote in full: *Occasionally it happens that we seek the solution under insufficient hypotheses or in an incorrect sense, and for this reason do not succeed. The problem then arises: to show the impossibility of the solution under the given hypotheses, or in the sense contemplated.*

Davis-Putnam-Robinson(1964) made great strides towards showing the problems was undecidable. Matiyasevich(1970) completed the proof. The results is often attributed to all four. After the result of Davis-Putnam-Robinson, Davis predicted that the problem would be solved by a young Russian mathematician. And indeed he was right.

One (likely unintentional) feature of this chapter in *Honor* is a focus on the role of randomness in research. In 1944 Post told Davis that he thought Hilbert's 10th problem cried out for an unsolvability proof and this began Davis's lifelong obsession with the problem. A more drastic example is in the next paragraph.

In their attempt to code Turing machines into polynomials, Robinson, Davis, and Putnam had been reading Number Theory books full of obscure facts. Matiyasevich had just read the third

[3]In Gerrmany, then and now, an academic often writes a Thesis 4 to 10 years after their PhD. This is needed for later employment.

edition of Nikolia Vorob'ev's *Fibonacci Numbers*. In that book, published in 1969, there was a new result about the divisibility of Fibonacci numbers. Matiyasevich used it to prove that if F_n^2 divides F_m then F_n divides m. This allowed him to solve Hilbert's 10th problem (building on work of Davis-Putnam-Robinson). Robinson had read that book, but *not the third edition!*

This chapter also discusses Julia Robinson's difficulties being a female mathematician and Yuri Matiyasevich experience in Communist Russia. The lives of Davis and Putnam are of course also discussed. I've heard the following said though its not in the book: people mostly thought H10 was solvable and would take Number Theory— hence it took outsiders— Julia Robinson (a female), Hillary Putnam (a Philosopher), Martin Davis(a Logician)— to look at the problem in a different way.

The solution (or perhaps anti-solution) to H10 actually showed that for any computably enumerable set [4] A there is a polynomial $p(x_1, \ldots, x_n, x)$ such that

$$A = \{a \mid (\exists a_1, \ldots, a_n)[p(a_1, \ldots, a_n, a) = 0].$$

Hence many familiar sets such as the primes, at least theoretically, have such a representation. The proof is constructive so one can actually find such polynomials; however, it is not clear if one wants to.

Honor notes that we can now code many problems, including the Riemann Hypothesis, into polynomials. The book goes on to claim that (to paraphrase) *if H10 was solvable then we could solve all of these problems.* If this remark was made in 1965 then it would reflect what people thought about complexity at the time (they didn't). But this book was written in 2002 when P, NP, and issues of complexity are well known. Even if H10 was solvable it may be that the algorithm to solve it takes so long to run that it would not help us to solve anything. Many known decidable theories (e.g., S1S and S2S) are also know to be quite complex. A decision procedure for them is not practical.

In *Math Dev* Davis-Matiyasevich-Robinson have an article that explores the implications of the solution to H10. They give an explicit polynomial for the primes (due to Jones) and note that it is NOW possible to prove a number is prime with a bounded number of operations (this had been previously unknown). They also look at questions of what is the max number of variables needed for any computably enumerable set (13 when *Math Dev* was written, 9 now) and what is the max degree needed (4, though with more variables). These bounds are not known to be tight. Further implications of the solution to H10 can be found in the book *Hilbert's 10th problem* by Matiyasevich.

3 Other Parts of the Books

Both books include Hilbert's paper where he proposed the problems. *Honor* has some historical context about Hilbert. *Math Dev* has a long article on current open problems suggested by the contributors of the volume. It was written in 1974 so P vs NP is not on the list.

Honor has a very short summary of which problems have been solved. Since some of the problems are research programs this can be somewhat subjective. Nevertheless, it is a good guide. In his estimation the following are the only ones that have not been solved: (1) H6, axiomitize physics (2) H8, the Riemann Hypothesis, and (3) H16, which informally asks for descriptions of algebraic curves.

[4] Computable enumerable sets were once called Recursively Enumerable sets. Ask your grandparents why.

In the year 2000 The Clay institute posed seven problems (called *The Millennium problems*) and offered $1,000,000 each for them. H8 is one of them. The Poincare Conjecture was one of them but has been solved. Since there is nobody of the stature of Hilbert nowadays this seems to be the only way to get problems out there— money.

Neither book mentions these problems. Since *Math Dev* was written in 1974 that is quite reasonable. While *Honor* was published in 2002 it was likely written before 2000.

4 Opinion

Honor is an excellent book with many tidbits of information that brings these people alive. Mathematicians may think in an abstract world but they live in the real one. This book brings that home by examining the lives they lead. The chapters also do a pretty good job of describing the mathematics involved on a level that a layperson can understand.

Math Dev is a hard read but a rewarding one. For many of Hilbert's problems you can get a sense of the work done on them after they were solved.