Some CFL’s that really require Proof
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1 Introduction

We give CFG’s for \{w \mid n_a(w) = n_b(w)\} and \{w \mid n_a(w) \neq n_b(w)\}.

2 A CFG for \(L_{a's=b's} = \{w \mid n_a(w) = n_b(w)\}\)

Let \(G\) be the CFG:
\[
\begin{align*}
S & \rightarrow aSb \\
S & \rightarrow bSa \\
S & \rightarrow SS \\
S & \rightarrow e
\end{align*}
\]

Theorem 2.1 \(L_{a's=b's} = L(G)\).

Proof:
1) \(L(G) \subseteq L_{a's=b's}\).
   
   It is easy to show that if \(S \Rightarrow \alpha\) (and \(\alpha\) may have \(S\)'s in it) then \(n_a(\alpha) = n_b(\alpha)\) by an induction on the number of steps in the derivation. Hence any string in \(L(G)\) (so all terminals) is in \(L_{a's=b's}\).

2) \(L_{a's=b's} \subseteq L(G)\).

   We proof this by induction on \(|w|\). If \(|w| = 0\) then use \(S \Rightarrow e\).

   Assume that any string in \(L_{a's=b's}\) of length \(< n\) is in \(L(G)\). Let \(w \in L_{a's=b's}\) and \(|w| = n\). We show \(w \in L(G)\). There are two cases depending on if \(w\) begins with \(a\) or \(b\). We do the \(w\) begins with a case. The other case is similar. Let \(w = a\sigma_2 \cdots \sigma_n\).

   Find the least \(i\) such that \(a\sigma_2 \cdots \sigma_i \in L_{a's=b's}\).

   If \(i = n\) then \(\sigma_n = b\) so \(w = aub\) where \(u \in L_{a's=b's}\). Clearly \(|u| < |w|\) so \(u \in L(G)\) inductively. Use \(S \Rightarrow aSb\) and then \(S \Rightarrow u\) to get \(w = aub \in L(G)\).

   If \(i < n\) then let \(u_1 = a\sigma_2 \cdots \sigma_i\) and \(u_2 = \sigma_{i+1} \cdots \sigma_n\). Since \(u_1, u_2 \in L, |u_1| < |w|,\) and \(|u_2| < w,\) both \(u_1, u_2\) are in \(L(G)\) inductively. Use \(S \Rightarrow SS\) and \(S \Rightarrow u_1\) and \(S \Rightarrow u_2\) to get \(S \Rightarrow u_1u_2 = w\). 

3 A CFG for \(L_{a's\neq b's} = \{w \mid n_a(w) \neq n_b(w)\}\)

Note that
\[
L_{a's\neq b's} = \{w \mid n_a(w) < n_b(w)\} \cup \{w \mid n_a(w) > n_b(w)\}
\]

We show that
\[
L_{a's<b's} = \{w \mid n_b(w) < n_a(w)\}
\]
is a CFL. The proof that
\[
L_{a's>b's} = \{w \mid n_b(w) > n_a(w)\}
\]
is a CFL s similar. Then we just take the union.

Let $G$ be:

$G = S \rightarrow aT$
$S \rightarrow aS$
$S \rightarrow bSS$
$T \rightarrow aTb$
$T \rightarrow bTa$
$T \rightarrow TT$
$T \rightarrow e$

Note that the last four rules are our grammar for $\{ w \mid n_a(w) = n_b(w) \}$ from the last section. Hence we have the following:

**Lemma 3.1** \{ $w \mid T \Rightarrow w$ \} = $L_{a's=b's}$.

**Theorem 3.2** $L(G) = L_{a's<b's}$.

**Proof:**

1) $L(G) \subseteq L_{a's<b's}$.

We do this by the length of the derivation. If $S \Rightarrow w$ with a derivation of length 1 then the lemma is true vacously. If $S \Rightarrow w$ with a derivation of length 2 then the only one possible is $S \rightarrow aT$ and then $T \rightarrow e$ which yields $S \Rightarrow a$.

Assume that if $S \Rightarrow w'$ with a derivation of length $\leq n - 1$ then $w' \in L_{a's<b's}$. Let $S \Rightarrow w$ via a derivation of length $n$. We look at what the first rule could be.

If $S \rightarrow aT$ is the first rule then we don’t need the ind hyp. We know that $T$ yields strings $u \in L_{a's=b's}$ so $S \Rightarrow au \in L_{a's<b's}$.

If $S \rightarrow aS$ then note that $S \Rightarrow u$ by a derivation of length $\leq n - 1$. Hence inductively $u \in L_{a's<b's}$. Clearly $au \in L_{a's<b's}$.

If $S \rightarrow bSS$ then the two $S$’s both have derivations of length $\leq n - 1$ to strings. Let those strings be $u_1$ and $u_2$. Inductively $u_1, u_2 \in L_{a's<b's}$. Hence $bu_1u_2 \in LL$.

2) $L_{a's<b's} \subseteq L(G)$.

We prove this by induction on $|w|$. Since $w \in L_{a's<b's}$, $|w| \geq 1$. If $|w| = 1$ then $w = a$ which is easily derived.

Assume all strings in $L_{a's<b's}$ of length $\leq n - 1$ can be generated. Let $w \in L_{a's<b's}$ be of length $n$.

1. If $w = au$ where $u \in L_{a's=b's}$ then use $S \rightarrow aT$ and then $T \Rightarrow u$.
2. If $w = au$ where $u \in L_{a's<b's}$ then use $S \rightarrow aS$ and then $S \Rightarrow u$.
3. If $w = bu$ then note that $n_a(u) \geq n_b(u) + 2$. One can easily show that $u = u_1u_2$ where $u_1, u_2 \in L_{a's<b's}$. Use $S \rightarrow bSS$, $S \Rightarrow u_1$ and $S \Rightarrow u_2$.  

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