Converting a DFA to a REG EXP: An Example
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$M = (Q, \Sigma, \delta, s, F)$ is a DFA. $R(i, j, k)$ is a reg exp for $\{ x \mid \delta(i, x) = j \}$.

Recall:

\[ R(i, j, 0) = \{ \sigma \in \Sigma \mid \delta(i, \sigma) = j \}. \]

\[ R(i, i, 0) = \{ \varepsilon \} \cup \{ \sigma \in \Sigma \mid \delta(i, \sigma) = j \}. \]

\[ R(i, j, k) = R(i, j, k - 1) \cup R(i, k, k - 1)R(k, k, k - 1)^*R(k, j, k - 1) \]

The regular expression for the language accepted by $M$ is
\[ \bigcup_{f \in F} R(1, f, n) \]

We will look at the DFA on the other sheet. (NOTE: its actually an NDFA since state 2 has no transition on a $b$. The method still works.) 1 is the start state, 3 is the only final state.

We want to know $R(1, 3, 3)$. Rather than compute all $3 \times 3 \times 4 = 36$ $R(i, j, k)$’s, we see which ones we need.

ALL OF THE $R(-, -, 3)$ THAT WE NEED: $R(1, 3, 3)$. (only 1)

Since $R(1, 3, 3) = R(1, 3, 2) \cup R(1, 3, 2)R(3, 3, 2)^*R(3, 3, 2)$

ALL OF THE $R(-, -, 2)$ THAT WE NEED: $R(1, 3, 2), R(3, 3, 2)$. (only 2)

We need $R(1, 3, 2)$. We use

\[ R(1, 3, 2) = R(1, 3, 1) \cup R(1, 2, 1)R(2, 2, 1)^*R(2, 3, 1) \]

Hence we need $R(1, 3, 1), R(1, 2, 1), R(2, 2, 1), R(2, 3, 1)$.

We need $R(3, 3, 2)$. ANOTHER SHORTCUT: Since state 3 is a self-loop it cannot ever use any other state, so $R(3, 3, 2) = R(3, 3, 0)$. We keep this in mind for later.

ALL OF THE $R(-, -, 1)$ THAT WE NEED:

$R(1, 2, 1), R(1, 3, 1), R(2, 2, 1), R(2, 3, 1)$ (only 4).

We are not going to bother to figure out which $R(-, -, 0)$ we need since its easier to just computer all nine of them. Note that we will get $R(3, 3, 0)$ which we need.

We first look at ALL of the $R(i, j, 0)$.

\[ R(1, 1, 0) = \varepsilon \]
\[ R(1, 2, 0) = a \]
\[ R(1, 3, 0) = b \]
\[ R(2, 1, 0) = \emptyset \]
\[ R(2, 2, 0) = \varepsilon \]
\[ R(2, 3, 0) = a \]
\[ R(3, 1, 0) = \emptyset \]
\[ R(3, 2, 0) = \emptyset \]
\[ R(3, 3, 0) = \varepsilon \cup a \cup b \]
We now look at all of the $R(i, j, 1)$ that we need.

\[
R(1, 2, 1) = R(1, 2, 0) \cup R(1, 1, 0)R(1, 1, 0)^*R(1, 2, 0) = a \cup ee^*a = a
\]
\[
R(1, 3, 1) = R(1, 3, 0) \cup R(1, 1, 0)R(1, 1, 0)^*R(1, 3, 0) = b \cup ee^*b = b
\]
\[
R(2, 2, 1) = R(2, 2, 0) \cup R(2, 1, 0)R(1, 1, 0)^*R(1, 2, 0) = e \cup \emptyset e^*a = e \cup \emptyset = e
\]
\[
R(2, 3, 1) = R(2, 3, 0) \cup R(2, 1, 0)R(1, 1, 0)^*R(1, 3, 0) = a \cup \emptyset e^*b = e \cup \emptyset = a
\]

We now look at all of the $R(i, j, 2)$ that we need.

\[
R(1, 3, 2) = R(1, 3, 1) \cup R(1, 2, 1)R(2, 2, 1)^*R(2, 3, 1) = b \cup ae^*a = b \cup aa
\]
\[
R(3, 3, 2) = R(3, 3, 0) = e \cup a \cup b
\]

We now look at all of the $R(i, j, 3)$, just $R(1, 3, 3)$.

\[
R(1, 3, 3) = R(1, 3, 2) \cup R(1, 3, 2)R(3, 3, 2)^*R(3, 3, 2) = (b \cup aa) \cup (b \cup aa)(e \cup a \cup b)^*(a \cup b).
\]

Reg Exp for the language is $R(1, 3, 3)$ above.