HW 1 CMSC 452. Morally DUE Feb 7

NOTE- THIS HW IS THREE PAGES LONG!!!

SOLUTIONS

THROUGOUT THIS HW YOU CAN ASSUME:

- The union of a finite number of countable sets is countable.
- The union of a countable number of finite sets is countable (not quite true if they are all the same set, but avoid that case).
- The union of a countable number of COUNTABLE sets if countable.
- The cross product of a finite number of countable sets is countable.
- The following sets are countable: N, Z, Q.
- The following sets are uncountable: $(0, 1)$, R.

1. (0 points) READ UP ON COUNTABILITY ON THE WEB. READ MY NOTES ON THE *HARD* HIERARCHY- WHICH WILL BE AVAILABLE LATER. What is your name? Write it clearly. Staple your HW. When is the midterm? Where is the midterm? When is the Final? IMPORTANT- I WANT TO MAKE SURE I HAVE YOUR CORRECT EMAIL ADDRESSES. I HAVE EMAILED ALL OF YOU USING WHAT I CURRENTLY THINK IS YOUR EMAIL ADDRESS BUT IF YOU DIDN'T GET IT THEN EMAIL ME ASAP TO GIVE ME YOUR REAL EMAIL ADDRESS.

2. (40 points) For each of the following sets say if its is

   - Empty
   - Finite but not empty
   - Countable (this implies NOT finite)
   - Uncountable

   And EXPLAIN your answer.

   NOTE: Throughout this HW $\mathsf{N} = \{1, 2, 3, \ldots\}$ it does NOT include 0.

   (a) The set of all functions from N to $\{0, 1, 2\}$
   (b) The set of all functions from N to $\{0, 1\}$
   (c) The set of all functions from N to $\{0\}$
   (d) The set of all functions from N to $\emptyset$
   (e) The set of a function from N to N that are INCREASING (so $x < y$ implies $f(x) \le f(y)$).
   (f) The set of a function from N to N that are strictly INCREASING (so $x < y$ implies $f(x) < f(y)$).
   (g) The set of a function from N to N that are DECREASING (so $x < y$ implies $f(x) \ge f(y)$).
   (h) The set of a function from N to N that are strictly DECREASING (so $x < y$ implies $f(x) > f(y)$).

SOLUTION TO PROBLEM 2

(a) The set of all functions from N to $\{0, 1, 2\}$
ANSWER: UNCOUNTABLE. Assume Countable. Then there is a listing $f_1, f_2, \ldots$. Let

$$F(x) = f_x(x) + 1 \pmod 3$$

Clearly F maps N to $\{0, 1, 2\}$. By the usual arguments F is NOT on the list.

(b) The set of all functions from N to $\{0, 1\}$

ANSWER: UNCOUNTABLE. Assume Countable. Then there is a listing $f_1, f_2, \ldots$. Let

$$F(x) = f_x(x) + 1 \pmod 2$$

Clearly F maps N to $\{0, 1\}$. By the usual arguments F is NOT on the list.

(c) The set of all functions from N to $\{0\}$

ANSWER: FINITE but NOT EMPTY. The ONLY such function is the constant function $f(x) = 0$.

(d) The set of all functions from N to $\emptyset$

ANSWER: EMPTY. There are no such functions. You can't map (say) 1 anywhere.

(e) The set of a function from N to N that are INCREASING (so $x < y$ implies $f(x) \le f(y)$).

ANSWER: UNCOUNTABLE Assume Countable. Then there is a listing $f_1, f_2, \ldots$. Let

$$F(1) = f_1(1) + 1$$

$$(\forall n \ge 2)[F(n) = F(n-1) + f_n(n) + 1]$$

Clearly F is increasing. In fact, its increasing so we can use the same argument in the next question. Note that $F(n) \ne f_n(n)$, so by the usual arguments F is not no the list.

(f) The set of a function from N to N that are strictly INCREASING (so $x < y$ implies $f(x) < f(y)$).

ANSWER: UNCOUNTABLE, Same proof as the last question.

(g) The set of a function from N to N that are DECREASING (so $x < y$ implies $f(x) \ge f(y)$).

ANSWER: COUNTABLE

We first show that the set is infinite: For each i the function $f(n) = i$ is in the set.

We now show that the set is the countable union of countable sets.

Note that:

$$f(1) \geq f(2) \geq f(3) \geq \cdots$$

We also know that all of the $f(n)$'s are ≥ 1. Hence the $f(i)$ are eventually constant. Formally

$$(\exists n_0, i)(\forall n \geq n_0)[f(n) = i]$$

We show the set is countable by showing it is a countable union of countable sets. Fix n_0, i. Let
$FUN_{n_0,i}$ be the set of decreasing functions f such that

$$(\forall n \geq n_0)[f(n) = i].$$

We show this set is countable.
Map each $f \in FUN_{n_0,i}$ to the ordered $(i-1)$-tuple

$$(f(1), f(2), \ldots, f(i-1))$$

This map is clearly a bijection. The co-domain is the set of all i-tuples of natural numbers that are decreasing and have all elements $\geq i$. We leave it to the reader to show that this set is countable.

(h) The set of a function from N to N that are strictly DECREASING (so $x < y$ implies $f(x) > f(y)$).
ANSWER: EMPTY. There are no such functions. Lets say f is in this set and $f(1) = a$. Then
$f(2) \leq a - 1$
$f(3) \leq a - 2$
$\vdots$
$f(a) \leq a - (a-1) = 1$
$f(a+1) \leq 0$
$f(a+2) \leq -1$
So f takes on negative values. But f is supposed to be from N to N.

3. (30 points) Let the $BILL_i$ numbers be defined as follows:

   - $BILL_0 = \mathsf{Q}$ (the rationals)
   - $BILL_{i+1}$ is the union of the following three sets:
     - $BILL_i$
     - $\{x + y : x, y \in BILL_i\}$
     - $\{x^y : x, y \in BILL_i\}$.

   Let $BILL = \bigcup_{i=0}^{\infty} BILL_i$.

   (a) Is $BILL$ countable or uncountable? Proof your result.

   (b) Let $BILL[x]$ be the set of polynomials with coefficients in $BILL$. Let $BILLBILL$ be the set of all roots of equations in $BILL[x]$. Is $BILLBILL$ countable or uncountable? Proof your result.

SOLUTION TO PROBLEM 3

3a) Countable. We show (formarlly by induction) that each $BILL_i$ is countable.

Base Case: $BILL_0 = \mathsf{Q}$ so that is countable.

Induction Hypothesis (IH): $BILL_i$ is countable.

$BILL_{i+1}$ is the union of three sets. We show that each one is countable.

$$A_1 = BILL_i$$

This is countable by the IH.

$$A_2 = \{x + y : x, y \in BILL_i\}$$

:

Since $BILL_i$ is countable, $BILL_i \times BILL_i$ is countable. List out this set of ordered pairs

$$(x_1, y_1), (x_2, y_2), \ldots$$

Then list out, though eliminate repeats:

$$x_1 + y_1, x_2 + y_2, \ldots$$

and you have a listing of A_1.

$$A_3 = \{x^y : x, y \in BILL_i\}$$

Similar to A_2.

3b) Countable. View a poly of degree d in $BILL[x]$ as an element of $BILL^{d+1}$. For example:

$$(\sqrt{7})x^2 - 3x + 7$$

is viewed as $(\sqrt{7}, -3, 7)$.

Since $BILL^d$ are all countable, the set $\bigcup_{d=2}^{\infty} BILL^d$ is countable, $BILL[x]$ is countable.

List out all polys in $BILL[x]$:

$$p_1, p_2, p_3, \ldots$$

WE ARE NOT DONE!!

List out, for each p_i, all of its roots, and eliminate duplicates. This gives you a listing of all roots of polys in $BILL[x]$.

4. (30 points) Show that $7^{1/3}$ does not satisfy any quadratic equation over the integers using the method shown in class.

SOLUTION TO PROBLEM 4

Assume, by way of contradiction, that there exists $a_2, a_1, a_0 \in \mathsf{Z}$ such that

$a_2 \times 7^{7/3} + a_1 \times 7^{1/3} + a_0 \times 1 = 0$ such that

We can assume that the gcd of a_2, a_1, a_0 is 1 since otherwise we could divide it out. In particular 7 does not divide all three of a_2, a_1, a_0.

Multiply the equation by $1, 7^{1/3}, 7^{7/3}$ to get

$a_2 \times 7^{7/3} + a_1 \times 7^{1/3} + a_0 \times 1 = 0$

$a_1 \times 7^{7/3} + a_0 \times 7^{1/3} + 7a_2 \times 1 = 0$

$a_0 \times 7^{7/3} + 7a_2 \times 7^{1/3} + 7a_1 \times 1 = 0$

We rewrite this as a matrix times a vector equals the 0 vector:

$$A = \begin{pmatrix} a_2 & a_1 & a_0 \\ a_1 & a_0 & 7a_2 \\ a_0 & 7a_2 & 7a_1 \end{pmatrix} \begin{pmatrix} 7^{7/3} \\ 7^{1/3} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since A times a non-zero vector gives 0, A must have det 0. Hence $A \pmod 7$ must have det 0:

$$A \pmod 7 = \begin{pmatrix} a_2 & a_1 & a_0 \\ a_1 & a_0 & 0 \\ a_0 & 0 & 0 \end{pmatrix}$$

If we expand this matrix on the last row we get that the det is a_0^3. Hence $a_0^3 \equiv 0 \pmod 7$, so $a_0 \equiv 0 \pmod 7$. Let $a_0 = 7b_0$. Now

$$A = \begin{pmatrix} a_2 & a_1 & 7b_0 \\ a_1 & 7b_0 & 7a_2 \\ 7b_0 & 7a_2 & 7a_1 \end{pmatrix}$$

Since A has det 0, so does A with the last col divided by 7. Hence the following matrix, B, had det 0:

$$B = \begin{pmatrix} a_2 & a_1 & b_0 \\ a_1 & 7b_0 & a_2 \\ 7b_0 & 7a_2 & a_1 \end{pmatrix}$$

Since B has det 0, so does $B \pmod 7$.

$$B \pmod 7 = \begin{pmatrix} a_2 & a_1 & b_0 \\ a_1 & 0 & a_2 \\ 0 & 0 & a_1 \end{pmatrix}$$

Expanding on the last row we get that the det of $B \pmod 7$ is $-a_1^3$. If $-a_1^3 \equiv 0 \pmod 7$ then $a_1 \equiv 0 \pmod 7$. Let $a_1 = 7b_1$. Hence

$$B = \begin{pmatrix} a_2 & 7b_1 & b_0 \\ 7b_1 & 7b_0 & a_2 \\ 7b_0 & 7a_2 & 7b_1 \end{pmatrix}$$

Since B has det 0, so does C which we obtain by dividing every elt of the middle col by 7:

$$C = \begin{pmatrix} a_2 & b_1 & b_0 \\ 7b_1 & b_0 & a_2 \\ 7b_0 & a_2 & 7b_1 \end{pmatrix}$$

Since C has det 0, so does $C \pmod 7$ which is:

$$C \pmod 7 = \begin{pmatrix} a_2 & b_1 & b_0 \\ 0 & b_0 & a_2 \\ 0 & a_2 & 0 \end{pmatrix}$$

The det of this by expandion on bottom row is a_2^3. Since $a_2^3 \equiv 0 \pmod 7$, $a_2 \equiv 0 \pmod 7$.

So we have a_0, a_1, a_2 are all divisible by 7. This contradicts a_0, a_1, a_2 being in lowest terms.