HW 7 CMSC 452. DUE DUE Mar 28

(NO late HW allowed since you have ALL of spring break AND I want to go over it on March 28 before the MIDTERM which is on March 30.)

THIS HOMEWORK IS TWO PAGES

SOLUTIONS

1. (5 points) What is your name? Write it clearly. When is the midterm? Write that clearly too. Staple your HW. WHAT IS THE DAY/TIME OF THE MIDTERM? (HINT: The Midterm is March 30 IN CLASS at 11:00.)

2. (30 points) Let

$$L = \{w : \#_a(w) \text{ is a cube }\}.$$

   (a) Show that L is NOT regular using clsoure properties and the Aaron George Pumping Lemma

   (b) Show that L is NOT regular using Communication Complexity.

SOLUTION TO PROBLEM TWO

2.a) Assume that L is regular. Then so is $L \cap a^* = \{a^{n^3} : n \in \mathsf{N}\}$ Assume the DFA is of of size s. Let n be large, we'll see how large later.

Let $w = a^{n^3}$. By the ELP $w = xyz$ such that $|xy| \le 2s$ and $xy^*z \in L$.

Let $x = a^{n_1}$, $y = a^{n_2}$, $z = a^{n_3}$. We know that $n_2 \neq 0$ and

$n_1 + n_2 + n_3 = n^3$.

$w = a^{n_1}a^{n_2}a^{n_3}$

SO $a^{n_1}a^{2n_2}a^{n_3} = a^{n_1+2n_2+n_3}$.

Hence $n_1 + 2n_2 + n_3$ is a cube.

Hence

$$n_1 + 2n_2 + n_3 \ge (n+1)^3 = n^3 + 3n^2 + 3n + 1$$

$$(n_1 + n_2 + n_3) + n_2 \ge n^3 + 3n^2 + 3n + 1$$

$$n^3 + n_2 \geq n^3 + 3n^2 + 3n + 1$$

$$n_2 \geq 3n^2 + 3n + 1$$

AH- but recall that $|xy| \leq 2s$ so $n_2 \leq 2s$.
Hence

$$2s \geq n_2 \geq 3n^2 + 3n + 1$$

NOW we know how big to take n: take $n = s$ (even smaller would do).
This leads to the contradiction:

$$2s \geq 3s^2 + 3s + 1$$

2.b) Using Communication Complexity
Let
$A_n = \{(x, y) : |x| = |y| = n \wedge xy \in L\}$
We will only be looking at
$A_{n^3} = \{(x, y) : |x| = |y| = n^3 \wedge xy \in L\}$
We produce, for every n, a fooling set for A_{n^3} of size n.
(a^{n^3}, b^{n^3})
$(a^{n^3-1}b, ab^{n^3-1})$
$(a^{n^3-2}b, a^2b^{n^3-2})$
$\vdots$
$(a^{n^3-i}b, a^ib^{n^3-i})$
$\vdots$
$(a^{n^3-n}b, a^nb^{n^3-n})$
All of the ordered pairs are in A_{n^3}. We want that if you take the LHS of any of them with the RHS of any other one the resulting pair is NOT in A_{n^3}. Let $i < j$.

$(a^{n^3-i}b, a^j b^{n^3-j})$

We want that

$n^3 - i + j$ is NOT a cube.

So we want

$n^3 - i + j < (n+1)^3 = n^3 + 3n^2 + 3n + 1$

$j - i < 3n^2 + 3n + 1$

What is the largest $j-i$ can be? The largest j can be is n. The smallest i can be is 0.

$$j - i \le n < 3n^2 + 3n + 1.$$

Therefore $n^3 - i + j$ is NOT a cube. Hence we have a fooling set.

3. (25 points) Let

$$L = \{a^p : p \text{ is prime }\}.$$

   (a) (5 points) Show that, for all natural numbers g, (‘g’ for ‘gap’) there are two consecutive primes that are more than g apart. (You will need this for the next two problems. If you can't solve this problem then solve the next two using it, and you will get full credit on the next two, though not on this one.)

   (b) (10 points) Show that L is NOT regular using the Aaron George Pumping Lemma.

   (c) (10 points) Show that L is NOT regular using Communication Complexity.

SOLUTION TO PROBLEM THREE

3.b) Assume that L is regular. Then $L \cap a^* = \{a^p : p \text{ is prime}\}$ is regular. Assume the DFA is of of size s. Let n be large, we'll see how large later.

Let $w = a^p$. By the ELP $w = xyz$ such that $|xy| \le 2s$ and $xy^*z \in L$.

Let $x = a^{n_1}$, $y = a^{n_2}$, $z = a^{n_3}$. We know that $n_2 \neq 0$ and

$n_1 + n_2 + n_3 = p$

$w = a^{n_1}a^{n_2}a^{n_3}$

SO $a^{n_1}a^{2n_2}a^{n_3} = a^{n_1+2n_2+n_3}$.

Hence $n_1 + 2n_2 + n_3$ is a prime.

Hence, letting p^+ be the next prime after p,

$$n_1 + 2n_2 + n_3 \geq p^+$$

$$(n_1 + n_2 + n_3)n_2 \geq p^+$$

$$p + n_2 \geq p^+$$

$$n_2 \geq p^+ - p$$

AH- but recall that $|xy| \leq 2s$ so $n_2 \leq 2s$.

Hence

$$2s \geq n_2 \geq p^+ - p$$

NOW we know how pick p: Take a value of p such $p^+ - p > 2s$. (NOTE- the issue is not that p is large, since there likely an infinite number of primes p such that p and $p+2$ are prime. But its a matter of picking SOME prime p with large distance to next prime.)

This leads to the contradiction:

$$2s \geq p^+ - p$$

3.c) Using Communication Complexity

Assume that L is regular by a DFA on s states. Let

$A_n = \{(x, y) : |x| = |y| = n \wedge xy \in L\}$

We intentionally rename this with p for prime.

$A_p = \{(x, y) : |x| = |y| = p \wedge xy \in L\}$

If p is a prime then let p^+ be the next prime. We will find, for all primes p a fooling set of A_p of size $p^- p$. Since there are primes that are arb large distances to the next prime, this will suffice.

(a^p, b^p)

$(a^{p-1}b, ab^{p-1})$

$(a^{p-2}b, a^2b^{p-2})$

$\vdots$

$(a^{p-i}b, a^ib^{p-i})$

$\vdots$

$(a^{p-(p^+-p-1)}b, a^{p^+-p-1}b^{p-(p^+-p-1)})$

All of the ordered pairs are in A_p. We want that if you take the LHS of any of them with the RHS of any other one the resulting pair is NOT in A_p. Let $i < j$.

$(a^{p-i}b, a^jb^{p-j})$

We want that

$p - i + j$ is NOT a prime

So we want

$p - i + j < p^+$

$j - i < p^+ - p$

What is the largest $j - i$ can be? The largest j can be is $p^+ - p - 1$. The smallest i can be is 0. Hence

$$j - i \le p^+ - p - 1.$$

Therefore $p - i + j \le p^+ - 1$ is NOT a prime.

4. (40 points) Let SQ be the set of squares, CU be the set of cubes, PR be the set of primes.

   We know from class and from this HW that the following sets ae not regular:

   $\{w : \#_a(w) \in SQ\}$

   $\{w : \#_a(w) \in CU\}$

   $\{w : \#_a(w) \in PR\}$

   THINK ABOUT: What property of SQ, CU, PR did we use in the proof?

   And now the real problem: Find a property of sets PROP so that the following is true, and PROVE it:

   If $B \subseteq \mathsf{N}$ has property PROP then

   $$\{w : \#_a(w) \in B\}$$

   is not regular. (HINT- I used Comm Complexity. There might be a proof using Pumping, I don't know.)

   SOLUTION TO PROBLEM 4

   PROP: there are arbitrary large gaps between elements.

   Proof omitted. But note that in the proofs for SQ, CU, PR that's ALL we needed.