**HW 10 CMSC 452. Morally Due April 25**
**THIS HW IS TWO PAGES!**
**SOLUTIONS**

1. (0 points BUT if you don't do it you'll get a 0 on the entire HW) What is your name? Write it clearly. Staple the HW.

2. (20 points) Prove that the following language is NOT context free:

$$\{a^n b^{2n} c^{3n} : n \in \mathsf{N}\}$$

**SOLUTION TO PROBLEM TWO**

This is a standard Pumping Lemma argument so I'll skip it.

3. (20 points) Let $G$ be the following context free grammar:

$S \to e$

$S \to SS$

$S \to bSaa$

$S \to aSab$

$S \to aSba$

$S \to aaSb$

$S \to abSa$

$S \to baSa$

$S \to aSbSa$

   (a) (5 points) Write down 10 strings from $L(G)$. (You do NOT need to show work.)
   (b) (5 points) Give a conjecture as to what $L(G)$ is. (Call this $L'$)
   (c) (10 points) Prove the easy half of your conjecture: That $L(G) \subseteq L'$.
   (d) Extra Credit - prove the hard half.

**SOLUTION TO PROBLEM THREE**

3.a) e.g. *baa*, *aab*, *aba*, *aabaab*, *abaaba*.

3.b) Conjecture: $\{w : \#_a(w) = 2\#_b(w)\}$

3.c) Let

$L = \{w : \#_a(w) = 2\#_b(w)\}$

$L(G) \subseteq L$:

We need to prove something a bit stronger:

Let $L'(G)$ be the set of ALL strings generated by $G$, even ones that have $S$ in them.

**Claim:** If $w \in L'(G)$ then $\#_a(w) = 2\#_b(w)$ (and we have no comment on $\#_S(w)$.

**Proof of Claim:** We prove this by induction on the number of times a rule is applied.

Base Case: Look at the rules! They all produce strings with twice as many $a$'s and $b$'s.

IH: If $w$ is produced with $n-1$ rules then $\#_a(w) = 2\#_b(w)$.

IS: Assume $w$ is produced with $n$ rules. Then after $n-1$ rules we have $w_1$ and then one more rule gets you to $w$. Since a rule was applied there must be an $S$ in $w_1$. By the IH $w_1$ has twice as many $a$'s as $b$'s. Note that all of the rules applied to an $S$ in $w_1$ would preserve this.

END OF PROOF OF CLAIM.

4. (30 points)

   (a) (15 points) Show that the set

$$\{G : G \text{ has a clique of size } 17\}$$

   is in P.

   (You can assume the input is an adj matrix. This goes for part b of this problem, and also the next problem.)

(b) (15 points) Let $k \in \mathsf{N}$. Show that the set

$$\{G : G \text{ has a clique of size } k\}$$

is in P.

(c) (0 points but please think about) The algorithm you got for the last problem had $k$ in the exponent (this is still poly time since $k$ is a constant). Do you think you can do this with $\sqrt{k}$ in the exponent? Some smaller function of $k$?

**SOLUTION TO PROBLEM FOUR**

4.a) Here is the algorithm: Look at ALL 17-sized subsets of $V$. For each one check if it is a clique. Checking takes $O(1)$ steps since you are looking at just $\binom{17}{2}$ entries in the matrix. There are $\binom{n}{17} \le O(n^{17})$ subsets to look at. Hence the algorithm takes $O(n^{17})$ steps.

4.b) Here is the algorithm: Look at ALL $k$-sized subsets of $V$. For each one check if it is a clique. Checking takes $O(1)$ steps since you are looking at just $\binom{k}{2}$ entries in the matrix. There are $\binom{n}{k} \le O(n^k)$ subsets to look at. Hence the algorithm takes $O(n^{17})$ steps.

5. (30 points) RECALL: If $G = (V, E)$ is a graph then *a vertex cover of size* $k$ is a set of vertices $U \subseteq V$ such that $(\forall (x, y) \in E)[x \in U \vee y \in U]$ (so every edge has at least one vertex in $U$).

(a) (15 points) Show that the set

$$\{G : G \text{ has a vertex cover of size } 17\}$$

is in P.

(b) (15 points) Let $k \in \mathsf{N}$. Show that the set

$$\{G : G \text{ has a vertex cover of size } k\}$$

is in P.

(c) (0 points but please think about) The algorithm you got for the last problem had $k$ in the exponent (this is still poly time since $k$ is a constant). Do you think you can do this with $\sqrt{k}$ in the exponent? Some smaller function of $k$?

**SOLUTION TO PROBLEM FIVE**

5.a) Here is the algorithm: Look at ALL 17-sized subsets of $V$. For each one check if it is a VC. Checking takes $O(|E|) - = O(n^2)$ steps since you are looking at every edge and seeing if some element of $U$ is in it. There are $\binom{n}{17} \le O(n^{17})$ subsets to look at. Hence the algorithm takes $O(n^{17}n^2) = O(n^{19})$ steps.

5.b) Similar to 5a.

6. (0 points, but Think About) Let

$$VC = \{(G,k) : \ G \text{ has a Vertex Cover of size} \le k \ \}$$

Let

$$F_{VC}(G) = \text{ the size of the smallest Vertex cover of } G$$

Show that IF $VC \in P$ then $F_{VC} \in FP$ (a function that can be computed in poly time).

**SOLUTION TO PROBLEM SIX**

Note that the smallest Vertex Cover is of some size between 1 and $n$. First ask $VC$ $(G, n/2) \in VC$? Then do binary search. So this take $\log n$ times the poly for $VC$.