HW 11 CMSC 452. Morally Due May 2
THIS HW IS TWO PAGES!
SOLUTIONS

1. (0 points BUT if you don't do it you'll get a 0 on the entire HW) What is your name? Write it clearly. Staple the HW.

2. (20 points)

   Format (the student): Turing Machines should have the freedom to move L or R AND print!

   Bill (the professor): Rather than argue the point I'll make a problem set out of it!

   (a) DEFINE a Turing machine that can, on seeing an input and of course knowing its state, will change its state (or not) and: either (1) Move L and print a character, (2) Move R and print a character, (3) Stay at the same position and print a character. (You need only tell me what the δ function does- what is its domain and range.)

   (b) In the proof of Cook's theorem we needed to, for every type of instruction, have a formula. With these NEW Turing Machines do the following: For the instruction that on input a, in state p goes to state q, moves L and prints a b, give the formula that captures that. Do not worry about The formula will use variables of the form $x_{i,j,}$ with the third part being either an element of Σ or of $\Sigma \times Q$.

   SOLUTION TO PROBLEM TWO

   2a)

$$\delta : Q - \{h\} \to Q \times (L \times \Sigma \cup R \times \Sigma \cup S \times \Sigma)$$

   (S means Stand still, DO NOT MOVE.)

   2b) So we have $\delta(p, a) = (q, L, b)$

$$x_{i,j,(q,a)} \to x_{i+1,j-1,(p,b)} \land x_{i+1,j,a}$$

3. (20 points) 3) Let

$$COL = \{(G,k) : G \text{ is } k\text{-colorable }\}$$

Let

$$FCOL(G) = \text{ [the least } k \text{ such that } G \text{ is } k\text{-colorable]}$$

Show that if $COL \in P$ then $FCOL \in FP$ (functions computable in poly time).

How many queries to COL did your algorithm make?

SOLUTION TO PROBLEM THREE

This is just binary search.

First ask $(G, n/2) \in COL$, if YES then we know that $FCOL(G) \le n/2$. If not then we know that $n/2 \le FCOL(G) \le n$. Keep cutting the interval in half. So can find $FCOL$ in $\log n$ queries to COL.

4. (30 points) Let

$$COL = \{(G,k) : G \text{ is } k\text{-colorable }\}$$

(as before)

Let $FCOL2(G)$ be an actual optimal coloring. That is, the output is G together with a coloring of G with $FCOL(G)$ colors.

Show that if $COL \in P$ then $FCOL2 \in FP$.

How many queries to COL did your algorithm make?

SOLUTION TO PROBLEM FOUR

We give two solutions:

SOLUTION ONE:

First use COL to find $FCOL(G)$. So we know know k such that G is k-colorable.

KEY: Let H be the graph with two ends u, v such that in ANY k-coloring of H u and v are the same color. We can form H by using K_{k-1} and have both u and v connected to everything in it.

Let (G, u, v) be G with an H between u, v.

Ask $((G, 1, 2), k) \in COL$

Ask $((G, 1, 3), k) \in COL$

$\vdots$

Ask $((G, 1, \ell), k) \in COL$

Etc until you get a YES. Then KEEP H between 1 and ℓ gave a YES. We NOW know that the resulting graph is k-colorable AND 1 and ℓ have the same color. RENAME G as G with that device in it.

Now KEEP asking about

Ask $((G, 1, \ell + 1), k) \in COL$

Etc. until you get another one. Keep doing this At the end we have a set of vertices such that if they are all the same color as color 1 the graph is still k-colorable. Call this set ONE. Think of them as all colored k (you'll see why). KEY-

There IS a k-coloring where all the vertices in ONE are colored k.

There is NO k-coloring where all the vertices in ONE PLUS some other vertex are colored k.

SO we know that we can Color them all k, remove them, and the RESULTING graph is $k - 1$ colorable.

We then repeat the process.

We obtain a k coloring.

We ask about pretty much every ordered pair, so $O(n^2)$ queries.

SOLUTION TWO:

We know that COL is NP-complete. Hence ANY set in NP is $\le COL$.
Let $COLEXT =$

$$\{(G, k, \rho) \mid \ \rho \text{ is a partial } k\text{-coloring of } G \text{ that can be extended to a full } k\text{-coloring of } G\ \}$$

Since $COLEXT \in NP$, $COLEXT \le COL$.

We can now just keep guessing what to color the next node and ask $COLEXT$ (via asking COL) if it will work.

We ask for every node and every color if the current coloring can be extended. So $O(kn)$ queries.

5. (30 points) Consider the following statement:

There is a way to compute $FCOL2$ with $O(\log n)$ queries to COL.

Which of the following is true? Prove your result.

- KNOWN: the statement is TRUE.
- KNOWN: Assuming $P \neq NP$ the statement is TRUE.
- KNOWN: the statement is FALSE.
- KNOWN: Assuming $P \neq NP$ the statement is FALSE

SOLUTION TO PROBLEM FIVE

KNOWN: Assuming $P \neq NP$ the statement is FALSE

We show that if STATEMENT is TRUE then $FCOL \in FP$ hence $P = NP$.

KEY: if the algorithm runs in $O(\log n)$ queries then there are only $n^{O(1)}$ possible sequences of answers.

Simulate the algorithm by looking at ALL possible sequences of query-answers. This gives $n^{O(1)}$ different possible outputs. KEY1: each output is a possible optimal coloring of G. KE2: This is NOT that many possible query-answer sequences. For each one of them check that it IS a coloring and note how many colors it takes. The path that gives a CORRECT coloring with the MIN number of colors is the OPTIMAL coloring- output k the number of colors it uses. THAT'S $FCOL(G)$. Hence you can actually compute $FCOL(G)$. Hence $P = NP$.