HW 12 CMSC 452. Morally Due May 9
THIS HW IS TWO PAGES!
SOLUTIONS

THROUGHOUT THIS HW YOU MAY ASSUME:
3-COL is NP-complete
SAT is NP-complete.

1. (0 points BUT if you don't do it you'll get a 0 on the entire HW) What is your name? Write it clearly. Staple the HW.

2. (25 points) Let

$$COL_k = \{G \mid G \text{ is } k\text{-colorable } \}$$

   (a) Show that $COL_3 \le COL_4$.
   (b) Show that $COL_k \le COL_{k+1}$.
   (c) Show that $COL_4 \le COL_3$.

   SOLUTION TO PROBLEM TWO

   2a) Let G be a graph. let G' be G with one more node that is connected to ALL vertices

   G is 3-col IFF G' is 4-col.

   2b) Similarly to 2a

   2c) This is a dirty, stinking trick. $COL_4 \in NP$. COL_3 is NP-Complete. Hence $COL_4 \le COL_3$.

   IF we do this more carefully to try to really GET the reduction here is what you get:

   We know that $COL_4 \le SAT$ since SAT is NP-complete

   We know that $SAT \le COL_3$ since COL_3 is NP-complete.

   So $COL_4 \le SAT \le COL_3$.

   That reduction is INSANE! Is there a SANE reduction. Yes - in a paper of mine.

3. (25 points) Let

$$CLIQ1 = \{G : \ G \text{ has } n \text{ vertices and has a clique of size } n/3\}$$

$$CLIQ2 = \{G : \ G \text{ has } n \text{ vertices and has a clique of size } n/2\}$$

(Ignore divisibility issues for 2 and 3 dividing n.)

(a) Show that $CLIQ1 \le CLIQ2$

(b) Is either problem NP-complete? (HINT - look at the proof that $CLIQ$ is NP-complete carefully!)

SOLUTION TO PROBLEM THREE

3a) Let $G_1 \oplus G_2$ be the graph that is $G_1 \cup G_2$ and EVERY vertex in G_1 has an edge to EVERY vertex of G_2.

Let G be a graph. We want to map it to $G \oplus K_m$ for some m we need to determine.

If G has n vertices and a clique of size $n/3$ then

$G \cup K_m$ has $n + m$ vertices and a clique of size $n/3 + m$. So we need

$n/3 + m = (1/2)(n + m)$

$n/3 + m = n/2 + m/2$

$m/2 = n/6$

$m = n/3$.

SO $G' = G \oplus K_{n/3}$.

Then G has a clique of size $n/3$ IFF G' has a clique of size $(1/2)(n+n/3)$.

3b) The proof that 3-SAT $\le$ CLIQUE produces a G on $3n$ vertices (some n) where the $3n$ vertices are in n clumps of 3. We want a vertex with one vertex per clump, so a clique of size $n/3$. HENCE $CLIQ1$ is NP-complete. Since $CLIQ1 \le CLIQ2$, $CLIQ2$ is NP-complete.

4. (25 points) A formula is in *DNF FORM* of it is of the form $D_1 \vee \cdots \vee D_m$ where each D_i is the AND of literals.

$DNF - SAT$ is the set of DNF-formulas that are SATISFIABLE.

Show either, $DNF - SAT$ is NP-complete, or that $DNF - SAT$ is in P.

SOLUTION TO PROBLEM FIVE

DNF-SAT is in P.

Given $D_1 \vee \cdots \vee D_m$ ALL you need to do is make ALL of the literals in some D_i true. This is easy - if there is some D_i where you DO NOT have both a variable and its complement then you can make that D_i true and you're done. If ALL of the D_i's have a var and its compliment then CANNOT satisfy.

5. (25 points) Below is an algorithm for Vertex Cover of size k which has some [FILL THIS IN] in it. Your job: You guessed it!

   There is a global variable, I, in this recursive procedure.

   $VC(G, k)$

   (a) Remove all isolated vertices.

   (b) If there is any vertex v of degree $\geq k+1$ then v MUST go into the vertex cover because [FILL THIS IN]. So $I = I \cup \{v\}$. If $|I| \geq k+1$ then output NO and stop. Else let $G' = G - \{v\}$ and call $VC(G', k-1)$.

   (c) If there are no vertices of degree $\geq k+1$ then EVERY vertex is of degree $\leq k$. If there is a VC of size k then there are at most k^2 edges because [FILL THIS IN]. Hence there are at most $k^2 - 1$ vertices. By brute force you can solve this problem in time [FILL THIS IN].

   For our analysis we will assume that there is an algorithm that finds vertices of degree $\geq BLAH$ and removes them in time $O(n)$. We can just use n and later make the entire algorithm an O-of.

   The run time of this algorithm is [FILL THIS IN] because [FILL THIS IN].

SOLUTION TO PROBLEM FIVE

$VC(G, k)$

(a) Remove all isolated vertices.

(b) If there is any vertex v of degree $\geq k+1$ then v MUST go into the vertex cover because *if v does not go in then there are $k+1$ edges that must be dealt with by putting into the VC the other endpoint-*

that's $k+1$ *vertices, too many!*. So $I = I \cup \{v\}$. If $|I| \geq k+1$ then output NO and stop. Else let $G' = G - \{v\}$ and call $VC(G', k-1)$.

(c) If there are no vertices of degree $\geq k+1$ then EVERY vertex is of degree $\leq k$. If there is a VC of size k then there are at most k^2 edges because *we can count the edges as such: map every vertex in the VC to the set of edges it covers. There are* $\leq k$ *vertices in the VC, and each one covers* $\leq k$ *edges. So* $\leq k^2$ *edges.* Hence there are at most $k^2 - 1$ vertices. By brute force you can solve this problem in time $\binom{k^2-1}{k} \sim k^{2k}$.

For our analysis we will assume that there is an algorithm that finds vertices of degree $\geq BLAH$ and removes them in time $O(n)$.

Here is the analysis: Let $T(n,k)$ be the run time for n vertices, seeking VC of size k. Then

$$T(n,k) \leq n + \max\{T(n-1, k-1), k^{2k}\}$$

CLAIM: $T(n,k) \leq kn + k^{2k}$.

One can prove this by induction.