THROUGHOUT THIS HW YOU MAY ASSUME:
3-COL is NP-complete
SAT is NP-complete.

1. (0 points BUT if you don’t do it you’ll get a 0 on the entire HW) What is your name? Write it clearly. Staple the HW.

2. (25 points) Let \( \text{COL}_k = \{ G \mid \text{G is } k\text{-colorable} \} \)

(a) Show that \( \text{COL}_3 \leq \text{COL}_4 \).
(b) Show that \( \text{COL}_k \leq \text{COL}_{k+1} \).
(c) Show that \( \text{COL}_4 \leq \text{COL}_3 \).

SOLUTION TO PROBLEM TWO

2a) Let \( G \) be a graph. let \( G' \) be \( G \) with one more node that is connected to ALL vertices
\( G \) is 3-col IFF \( G' \) is 4-col.

2b) Similarly to 2a

2c) This is a dirty, stinking trick. \( \text{COL}_4 \in NP \). \( \text{COL}_3 \) is NP-Complete. Hence \( \text{COL}_4 \leq \text{COL}_3 \).

IF we do this more carefully to try to really GET the reduction here is what you get:

We know that \( \text{COL}_4 \leq \text{SAT} \) since SAT is NP-complete
We know that \( \text{SAT} \leq \text{COL}_3 \) since \( \text{COL}_3 \) is NP-complete.
So \( \text{COL}_4 \leq \text{SAT} \leq \text{COL}_3 \).

That reduction is INSANE! Is there a SANE reduction. Yes - in a paper of mine.
3. (25 points) Let

\[ \text{CLIQ}_1 = \{ G : \text{G has n vertices and has a clique of size } n/3 \} \]

\[ \text{CLIQ}_2 = \{ G : \text{G has n vertices and has a clique of size } n/2 \} \]

(Ignore divisibility issues for 2 and 3 dividing \( n \).)

(a) Show that \( \text{CLIQ}_1 \leq \text{CLIQ}_2 \)

(b) Is either problem NP-complete? (HINT - look at the proof that \( \text{CLIQ} \) is NP-complete carefully!)

**SOLUTION TO PROBLEM THREE**

3a) Let \( G_1 \oplus G_2 \) be the graph that is \( G_1 \cup G_2 \) and EVERY vertex in \( G_1 \) has an edge to EVERY vertex of \( G_2 \).

Let \( G \) be a graph. We want to map it to \( G \oplus K_m \) for some \( m \) we need to determine.

If \( G \) has \( n \) vertices and a clique of size \( n/3 \) then

\( G \cup K_m \) has \( n + m \) vertices and a clique of size \( n/3 + m \). So we need

\[
\frac{n}{3} + m = \left(\frac{1}{2}\right)(n + m)
\]

\[
\frac{n}{3} + m = \frac{n}{2} + \frac{m}{2}
\]

\[
\frac{m}{2} = \frac{n}{6}
\]

\[
m = \frac{n}{3}.
\]

SO \( G' = G \oplus K_{n/3} \).

Then \( G \) has a clique of size \( n/3 \) IFF \( G' \) has a clique of size \( (1/2)(n+n/3) \).

3b) The proof that 3-SAT \( \leq \text{CLIQUE} \) produces a \( G \) on \( 3n \) vertices (some \( n \)) where the \( 3n \) vertices are in \( n \) clumps of 3. We want a vertex with one vertex per clump, so a clique of size \( n/3 \). HENCE \( \text{CLIQ1} \) is NP-complete. Since \( \text{CLIQ1} \leq \text{CLIQ2} \), \( \text{CLIQ2} \) is NP-complete.

4. (25 points) A formula is in \( \text{DNF FORM} \) if it is of the form \( D_1 \lor \cdots \lor D_m \) where each \( D_i \) is the AND of literals.

\( \text{DNF} - \text{SAT} \) is the set of DNF-formulas that are SATISFIABLE.

Show either, \( \text{DNF} - \text{SAT} \) is NP-complete, or that \( \text{DNF} - \text{SAT} \) is in \( P \).
SOLUTION TO PROBLEM FIVE

DNF-SAT is in P.

Given $D_1 \vee \cdots \vee D_m$ ALL you need to do is make ALL of the literals in some $D_i$ true. This is easy - if there is some $D_i$ where you DO NOT have both a variable and its complement then you can make that $D_i$ true and you’re done. If ALL of the $D_i$’s have a var and its compliment then CANNOT satisfy.

5. (25 points) Below is an algorithm for Vertex Cover of size $k$ which has some [FILL THIS IN] in it. Your job: You guessed it!

There is a global variable, $I$, in this recursive procedure.

$VC(G, k)$

(a) Remove all isolated vertices.

(b) If there is any vertex $v$ of degree $\geq k + 1$ then $v$ MUST go into the vertex cover because [FILL THIS IN]. So $I = I \cup \{v\}$. If $|I| \geq k + 1$ then output NO and stop. Else let $G' = G - \{v\}$ and call $VC(G', k - 1)$.

(c) If there are no vertices of degree $\geq k + 1$ then EVERY vertex is of degree $\leq k$. If there is a VC of size $k$ then there are at most $k^2$ edges because [FILL THIS IN]. Hence there are at most $k^2 - 1$ vertices. By brute force you can solve this problem in time [FILL THIS IN].

For our analysis we will assume that there is an algorithm that finds vertices of degree $\geq BLAH$ and removes them in time $O(n)$. We can just use $n$ and later make the entire algorithm an O-of.

The run time of this algorithm is [FILL THIS IN] because [FILL THIS IN].

SOLUTION TO PROBLEM FIVE

$VC(G, k)$

(a) Remove all isolated vertices.

(b) If there is any vertex $v$ of degree $\geq k + 1$ then $v$ MUST go into the vertex cover because if $v$ does not go in then there are $k + 1$ edges that must be dealt with by putting into the VC the other endpoint-
that's $k+1$ vertices, too many! So $I = I \cup \{v\}$. If $|I| \geq k+1$ then output NO and stop. Else let $G' = G - \{v\}$ and call $VC(G', k-1)$.

(c) If there are no vertices of degree $\geq k + 1$ then EVERY vertex is of degree $\leq k$. If there is a VC of size $k$ then there are at most $k^2$ edges because we can count the edges as such: map every vertex in the VC to the set of edges it covers. There are $\leq k$ vertices in the VC, and each one covers $\leq k$ edges. So $\leq k^2$ edges. Hence there are at most $k^2 - 1$ vertices. By brute force you can solve this problem in time $(\binom{k^2-1}{k}) \sim k^{2k}$.

For our analysis we will assume that there is an algorithm that finds vertices of degree $\geq BLAH$ and removes them in time $O(n)$.

Here is the analysis: Let $T(n, k)$ be the run time for $n$ vertices, seeking VC of size $k$. Then

$$T(n, k) \leq n + \max\{T(n - 1, k - 1), k^{2k}\}$$

CLAIM: $T(n, k) \leq kn + k^{2k}$.

One can prove this by induction.