Closure of Regular Langs Under Union, Intersection, Complementation, and Projection
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1 Introduction

We give the constructions that show sketch the proof that all if \( L_1 \) and \( L_2 \) are regular and \( L_1 \cap L_2 \), \( L_1 \cup L_2 \), \( \overline{L} \), and \( proj(L) \) (which we will define) are regular.

**Def 1.1** A DFA is a tuple \((Q, \Sigma, \delta, s, F)\) where \( \delta : Q \times \Sigma \rightarrow Q \).

We define running a DFA \( M \) on a string \( x \) in the obvious way. If the DFA ends in a state in \( F \) then \( x \) is accepted. Otherwise its rejected.

2 The Construction for Intersection

**Theorem 2.1** If \( L_1 \) and \( L_2 \) are regular then \( L_1 \cap L_2 \) is regular.

**Proof:**
Let \( M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1) \) be the DFA for \( L_1 \). Let \( M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2) \) be the DFA for \( L_2 \).
We define the DFA for \( L_1 \cap L_2 \). Let \( M = (Q_1 \times Q_2, \Sigma, \delta, (s_1, s_2), F_1 \times F_2) \) where \( \delta \) is defines by, for \((q_1, q_2) \in Q_1 \times Q_2 \) and \( \sigma \in \Sigma \),

\[
\delta((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma)).
\]

The intuition is that the DFA \( M \) runs \( M_1 \) and \( M_2 \) at the same time. If both end up in \( F_1 \times F_2 \) then both \( M_1 \) and \( M_2 \) accepted.

3 The Construction for Union

**Theorem 3.1** If \( L_1 \) and \( L_2 \) are regular then \( L_1 \cup L_2 \) is regular.

**Proof:**
Let \( M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1) \) be the DFA for \( L_1 \). Let \( M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2) \) be the DFA for \( L_2 \).
We define the DFA for \( L_1 \cup L_2 \). Let \( M = (Q_1 \times Q_2, \Sigma, \delta, (s_1, s_2), F_1 \times Q_2 \cup Q_1 \times F_2) \) where \( \delta \) is defines by, for \((q_1, q_2) \in Q_1 \times Q_2 \) and \( \sigma \in \Sigma \),

\[
\delta((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma)).
\]

The intuition is that the DFA \( M \) runs \( M_1 \) and \( M_2 \) at the same time. If \( M_1 \) ends up in \( F_1 \) then we accept (independent of what \( M_2 \) does), and if \( M_2 \) ends up in \( F_2 \) then we accept (independent of what \( M_1 \) does).

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4 The Construction for Complementation

Theorem 4.1 If \( L \) is regular then \( \overline{L} \) is regular.

Proof: Let \( M = (Q, \Sigma, \delta, s, F) \) be the DFA for \( L \).
We define the DFA for \( \overline{L} \). Let \( M' = (Q, \Sigma, \delta, s, Q - F) \) (recall that \( Q - F = \{q \mid q \in Q \land q \notin F\} \).
The intuition is that the DFA \( M' \) runs \( M \) but does the opposite when it comes to accepting.

5 The Construction for Complimentation

To Compliment a DFA you say

\[ \text{My DFA, you lovely you look.} \]

6 The Construction for Nondeterminism

Recall the definition of an NDFA:

Def 6.1 An NDFA is a tuple \((Q, \Sigma, \Delta, s, F)\) where \( \Delta : Q \times (\Sigma \cup e) \to 2^Q \). (Recall that \( 2^Q \) is the powerset of \( Q \).

We DO NOT define running an NDFA \( M \) on a string \( x \). Instead we say that an NDFA accepts \( x \) if SOME way of running the machine ends up in a state in \( F \).

Theorem 6.2 If \( L \) is accepted by an NDFA then there exists a DFA such that accepts \( L \).

Proof: Let \( M = (Q, \Sigma, \Delta, s, F) \) be the NDFA for \( L \).
We define the DFA for \( L \). Let \( M' = (2^Q, \Sigma, \delta, s, F) \) where for \( A \in 2^Q \) and \( \sigma \in \Sigma \),

\[ \delta(A, \sigma) = \bigcup_{q \in A} \Delta(e^aqe^b, \sigma) \]

(The \( e^a \) and \( e^b \) are strings of the empty string.)

\[ F = \{ A \mid A \cap F \neq \emptyset \} \]

The intuition is that the DFA \( M' \) runs ALL possibilities for \( M \). If SOME possibility ends up accepting, then accept.
7 Closure under Projection

Notation 7.1 Let $\Sigma = \{0, 1\}^n$. Note that each element of $\Sigma$ is itself a string of $n$ bits. If $x \in \Sigma^*$ then $\text{proj}(x)$ is what you get by taking each symbol in $x$ and chopping off the last bit. So if $x \in (\{0, 1\}^n)^*$ then $\text{proj}(x) \in (\{0, 1\}^{n-1})^*$. If $L \subseteq (\{0, 1\}^n)^*$ then

$$\text{proj}(L) = \{\text{proj}(x) \mid x \in L\}.$$ 

Theorem 7.2 If $L$ is regular then $\text{proj}(L)$ is regular.

Proof: Let $M = (Q, (\{0, 1\}^n), \delta, s, F)$ be the DFA for $L$. We define an NDFA for $L$. Let $M' = (Q, \{0, 1\}^{n-1}, \Delta, s, F)$. For $q \in Q$ and $\sigma \in \{0, 1\}^{n-1}$

$$\Delta(q, \sigma) = \{\delta(q, \sigma0), \delta(q, \sigma1)\}.$$