DETERMINING IF $X = Y$

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OUR PROBLEM- EQ

1. Alice has x, Bob has y.
2. They want to see if $x = y$ communicating as few bits as possible.
3. We call this problem EQ.

OBVIOUS PROTOCOL

1. Alice has $a_1 \cdots a_n$. Bob has $b_1 \cdots b_n$.
2. Alice sends $a_1 \cdots a_n$ to Bob (n bits).
3. Bob compares $a_1 \cdots a_n$ to $b_1 \cdots b_n$.
   If equal send 1, else send 0. (1 bit.)

So EQ can be solved with $n+1$ bits.

VOTE!

1. EQ **REQUIRES** $\sim n$ bits.
2. Can do EQ **with** $\sim \sqrt{n}$ bits, but no better.
3. Can do EQ **with** $\sim \log n$ bits, but no better.
4. Stewart/Colbert **in** 2016.

BAD NEWS

EQ **REQUIRES** $n+1$ bits.
So, for Alice and Bob to determine if two n-bit strings are equal **REQUIRES** $n+1$ bits.
(Proven by Andrew Yao in 1979.)

ALLOW ERROR

What if we

1. Allow Alice and Bob to flip coins, and
2. allow a probability of error $\leq \frac{1}{n}$.

NAIVE IDEA

1. Alice has $a_1 \cdots a_n$. Bob has $b_1 \cdots b_n$.
2. Alice rand $S \subseteq \{1, \ldots, n\}$, $|S| = 10$.
3. For $i \in S$ Alice sends (i, a_i). $10 \log n$ bits.
4. For each (i, a_i) that Bob checks "$a_i = b_i$?".
5. If always YES, Bob sends 1, else sends 0.

GOOD AND BAD

1. Protocol is $\sim \log n$ bits. **GOOD!**
2. Prob of error $\rightarrow 1$ as $n \rightarrow \infty$. **BAD!**
3. Does well if input is unif chosen. **GOOD!**
4. Not really what we want. **BAD!**
5. KEY PROBLEM: Protocol too local.

LESS NAIVE IDEA

1. Alice has $a_1 \cdots a_n$. Bob has $b_1 \cdots b_n$.
2. Alice computes $a_1 + \cdots + a_n$.
   Sends $PAR(a_1 + \cdots + a_n) = 1$ if sum is **ODD**
   Sends $PAR(a_1 + \cdots + a_n) = 0$ if sum is **EVEN**.
3. Bob computes $PAR(b_1 + \cdots + b_n)$.
   $PAR(a_1 + \cdots + a_n) = PARITY(b_1 + \cdots + b_n)$ then 1 $(x = y)$
   $PAR(a_1 + \cdots + a_n) = PARITY(b_1 + \cdots + b_n)$ then 0 $(x \neq y)$

GOOD AND BAD

1. Only send ~ 1 bit. **GOOD.**
2. Bit used ALL of $a_1 \cdots a_n$. **GOOD.**
3. Protocol will be wrong alot. **BAD.**
4. Speculation: Can we use $a_n + \cdots + a_1$ remainder when divided by 3? 4? 5?

NEED MOD CONCEPT

$a \equiv b \pmod{c}$ means

1. a/c and b/c have same remainder.
2. $b \in \{0, 1, \ldots, c-1\}$.

EXAMPLES:

1. Any odd number is $\equiv 1 \pmod{2}$.
2. $100 \equiv 9 \pmod{7}$ since $100 = 7 \times 13 + 9$

NEED TO WORK MOD m

$Z_m = \{0, 1, \ldots, m-1\}$ and all of the operations are mod m.

1. Everyday Example: Clock Arithmetic
2. For all m you can do $+$, $-$, $\times$ in Z_m.
3. For m prime you can also do $\div$ in Z_m.

NEED A THEOREM

1. If f is a polynomial **over the reals** of degree d then f has at most d roots.
2. If f is a polynomial **over the complex numbers** of degree d then f has at most d roots. (d if you count multiplicities.)
3. Let p be a prime. If f is a polynomial **over Z_p** of degree d then f has at most d roots.

RANDOMIZED PROTOCOL

1. Alice has $a_0a_1\cdots a_{n-1}$. Bob has $b_0b_1\cdots b_{n-1}$.
   Alice sends Bob a prime p, $n^2 \le p \le 2n^2$.
2. Alice picks $z \in \{1,\ldots,p-1\}$ **RAND.**
   Alice computes, mod p,
   $y = a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1}$
   Alice sends (z,y) to Bob.
3. Bob computes, mod p,
   $y' = b_0 + b_1z + b_2z^2 + \cdots + b_{n-1}z^{n-1}$
   If $y = y'$ then send 1, else send 0.

GOOD!

1. Protocol exchanges $\sim \log n$ bits.
2. Prob of error is $\leq \frac{1}{n}$.
   WHY: If there is an error then z
   is a root of the poly $a(x) - b(x)$
   There are only n such roots so the probability
   of this is very low.
3. This result is due to Melhorn and Schmidt, 1982.

FOR MORE INFORMATION

COMMUNICATION COMPLEXITY
by
Kushilevitz and Nisan.