HW 6, CMSC 452. Morally DUE Mar 27

This HW is 200 points and counts twice as much as other HW

This HW is THREE PAGES LONG

(To make the midterm SHORTER, and to give you a break, there was NOT a question on countability. So, this is an additional HW just on that.)

QUESTION ONE (15 points each for 75 points total)

For each of the following sets, say if the set is:

FINITE (note: the empty set is countable)
COUNTABLE (that is, there is a bijection to N)
UNCOUNTABLE

(Note that a function must map **every** element of its domain.)
AND PROVE YOUR ANSWER.

1. The set of INCREASING functions from N to SQUARES. (A function f is INCREASING if $x < y$ implies $f(x) < f(y)$.)

   SOLUTION TO ONE.1

   (65 points)

   Uncountable.

   Assume, by way of contradiction, that the set is countable. Let

   $$f_1, f_2, f_3, \ldots$$

   We will create a function that IS increasing from N to SQUARES. but is NOT on the list. Let

   $g(i)$ = the least number that (1) LARGER THAN all elements of $\{g(1), \ldots, g(i-1)\}$, (2) is NOT $f_i(i)$, and (3) IS a square.

   $g(i)$ is clearly increasing from N to SQUARES.

   For all i, g is NOT f_i since $g(i) \neq f_i(i)$.

   END OF SOLUTION TO ONE.1

2. The set of INCREASING functions from SQUARES to N. (A function f is INCREASING if $x < y$ implies $f(x) < f(y)$.)

   SOLUTION TO ONE.2

Uncountable.

Assume, by way of contradiction, that the set is countable. Let

$$f_1, f_2, f_3, \ldots$$

We will create a function that IS increasing from SQUARES to N to SQUARES. but is NOT on the list. Let

$g(i^2)$ = the least number that (1) is LARGER than everything in $\{g(1), \ldots, g(i-1)\}$, (2) is NOT $f_i(i)$.

For all i, g is NOT f_i since $g(i^2) \neq f_i(i^2)$.

END OF SOLUTION TO ONE.2

3. The set of DECREASING functions from N to N (A function f is DECREASING if $x < y$ implies $f(x) > f(y)$.)

SOLUTION TO ONE.3

Finite.

Actually empty- there are not such functions. If $f(1) = 18$ then $f(2) \leq 17$, …, f has to be negative.

END OF SOLUTION TO ONE.3

4. The set of DECREASING functions from N to Z (A function f is DECREASING if $x < y$ implies $f(x) > f(y)$.)

SOLUTION TO ONE.4

Uncountable.

Assume, by way of contradiction, that the set is countable. Let

$$f_1, f_2, f_3, \ldots$$

We will create a function that IS deceasing from N to Z but is NOT on the list. Let

$g(i)$ = the largest INTEGER that (1) LESS THAN everything in $\{g(1), \ldots, g(i-1)\}$, (2) is NOT $f_i(i)$.

$g(i)$ is clearly increasing from N to SQUARES.

For all i, g is NOT f_i since $g(i) \neq f_i(i)$.

END OF SOLUTION TO ONE.4

5. (For this homework, a function f from N to N is *kruskalian* if $x < y$ implies $f(x) \geq f(y)$ (NOTE $\geq$ NOT $>$).) The set of kruskalian functions from N to N.

SOLUTION TO ONE.5

Countable.

If $f(1) = 18$ then $f(2), f(3), \ldots$ will have to be eventually constant. Hence every kruskalian function can be represented by an element of

$$(\mathsf{N} \cup \mathsf{N} \times \mathsf{N} \cup \mathsf{N} \times \mathsf{N} \times \mathsf{N} \cup \cdots) \times \mathsf{N}$$

The first part is the initial segment of the function, and the second is the constant that the function eventually is.

For example if

$f(1) = 19$

$f(2) = 19$

$f(3) = 16$

$f(4) = 10$

$f(5) = 3$

$f(6) = 3$

and for all $n \geq 7$ $f(n) = 2$

then this function is represented by $((19, 19, 16, 10, 3, 3), 2)$

So there is a bijection between kruskalian functions and

$$(\mathsf{N} \cup \mathsf{N} \times \mathsf{N} \cup \mathsf{N} \times \mathsf{N} \times \mathsf{N} \cup \cdots) \times \mathsf{N}$$

This set is countable since

(1)

$$\mathsf{N}, \mathsf{N} \times \mathsf{N}, \mathsf{N} \times \mathsf{N} \times \mathsf{N}, \ldots$$

are all countable, and the countable union of countable sets is countable.

and

(2) N is countable

and

(3) the cross product of two countable sets is co unable.

END OF SOLUTION TO ONE.5

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QUESTION TWO (65 points)

Consider the following proof that the rationals between 0 and 1 are uncountable. What is WRONG with the proof? (We ignore things like $.999\cdots = 1$, that is NOT the issue.)

Assume, by way of contradiction, that $Q \cap [0,1]$ is countable.

$$q_1, q_2, q_3, \ldots$$

be a listing of those rationals. We write them out with all of their digits:

$q_1 = .q_{11}q_{12}q_{13}\cdots$
$q_2 = .q_{21}q_{22}q_{23}\cdots$
$q_3 = .q_{31}q_{32}q_{33}\cdots$
$\vdots$

We will create a rational between 0 and 1 that is NOT on the list.

Let a hat $\hat{}$ over a number mean you add a 1 mod 10. so:

$\hat{0} = 1$
$\hat{8} = 9$
$\hat{9} = 0$.

The important thing is that $\hat{b} \ne b$.

We form the rational:

$$\hat{q_{11}}\hat{q_{22}}\hat{q_{33}}\cdots$$

This rational is NOT the ith on the list since it differs from q_i on the ith digit.

So the rationals between 0 and 1 are not countable.

WHAT IS WRONG WITH THIS PROOF?

SOLUTION TO PROBLEM TWO

The number produces is not on the list BUT it might not be a rational.

END OF SOLUTION TO PROBLEM TWO

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QUESTION THREE (60 points)

In your own words and pictures describe an algorithm that will:

Given a regular expression α, return an NFA that ACCEPTS exactly the strings that α GENERATES.

SOLUTION TO PROBLEM THREE

We do this by induction on the length of a regualr expression.

Base Case: Regex of length 1. This can only be e or some $\sigma \in \Sigma$. There is easily an NFA for this. Even a DFA.

IH: Assume that all regex of length $< n$ have NFA's.

IS: Let α be a regex of length n. *By the defintion of regex* α is of one of the following forms.

Case 1: $\alpha = \beta \cup \gamma$ where β and γ are regex. KEY: β and γ are of length $< n$. So by the IH they have NFA's. Use closure of NFA's under union.

Case 2: $\alpha = \beta\gamma$ where β and γ are regex. KEY: β and γ are of length $< n$. So by the IH they have NFA's. Use closure of NFA's under concat (NOTE- here you really want to use NFA's since closure of DFA's under concat is true, but messy).

Case 3: $\alpha = \beta^*$ where β is a regex. KEY: β is of length $< n$. So by the IH it has an NFA. Use closure of NFA's under $*$ (NOTE- here you really want to use NFA's since closure of DFA's under $*$ is true, but messy).

END OF SOLUTION TO PROBLEM THREE