HW 11 CMSC 452. Morally Due April 24
HW IS TWO PAGES!
SOLUTIONS

1. (5 points) What is your name? Write it clearly. Staple the HW.

2. (60 points) Let

$$IS = \{(G, k) : G \text{ has an independent set of size} \geq k\}$$

(A set of vertices, U, is an *Independent Set* if no vertex in U has an edge to any other vertex in U.)

   (a) Describe, in English — with pictures and an example — how you would, GIVEN a Boolean Formula $\phi(x_1, \ldots, x_n)$ produce a graph G and a number k such that:

$$\phi \in SAT \text{ iff } (G, k) \in IS$$

   (b) Write pseudocode for the procedure that takes a Boolean Formula ϕ and produces (G, k), as described above.

   (c) Apply your procedure to the Boolean Formula:

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2) \wedge x_3$$

   (d) If ϕ has L clauses and each clause has three variables, then how many vertices are in G and what is k?

   (e) Assume that 3-SAT is NP-complete. (See the next problem for definition of 3-SAT.) Find a function f such that:

$$ISF = \{G : G \text{ has } n \text{ vertices and an independent set of size} \geq f(n)\}$$

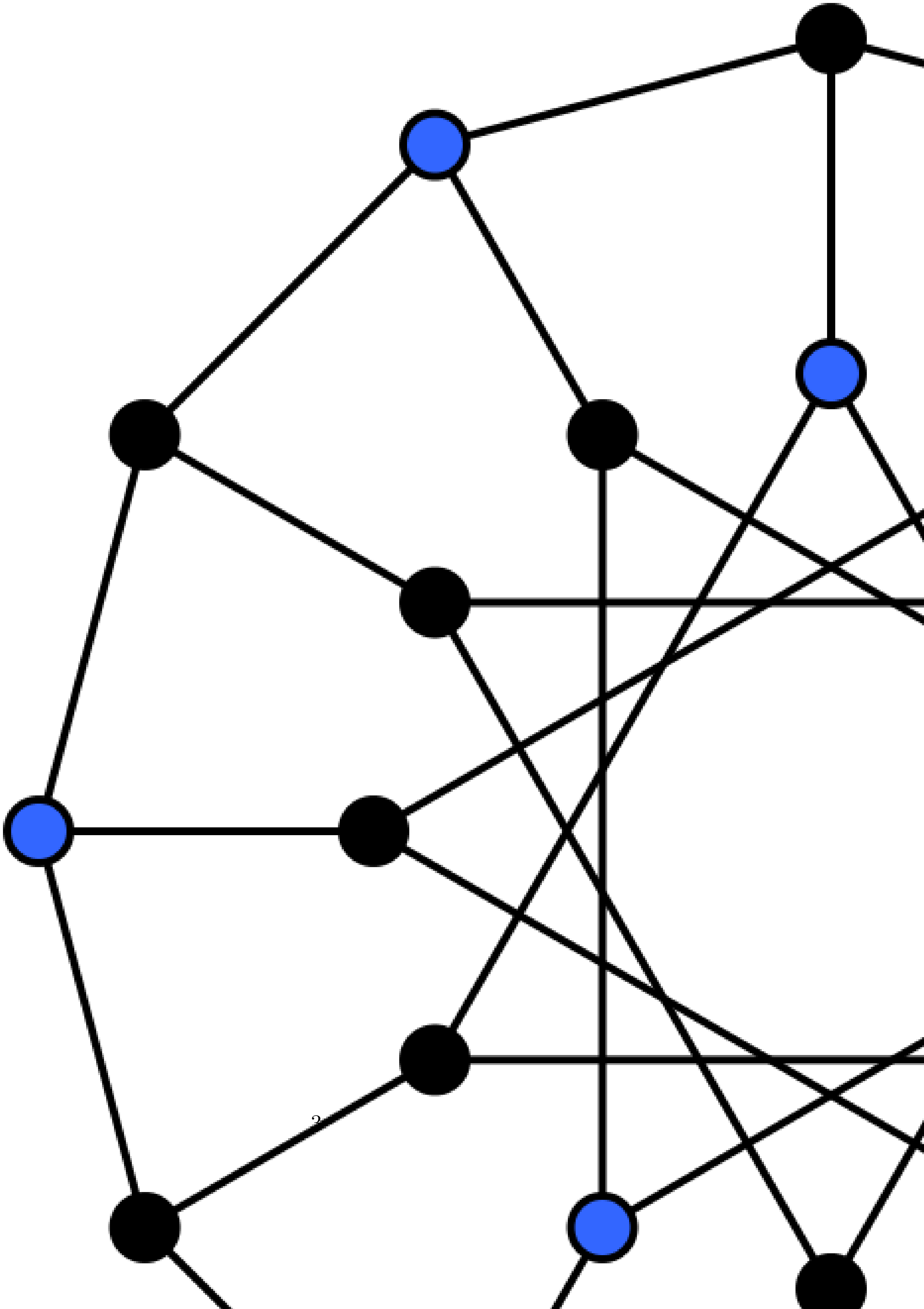
is NP-complete.

SOLUTION TO PROBLEM TWO

Suppose we have an instance of a boolean formula $C_1 \wedge C_2 \wedge \ldots$, where each C_i is the disjunction of 3 variables. Label the variables $x_1, \ldots x_n$ and their negations $\neg x_1, \ldots$. Create a graph G as follows:

For each variable in each clause, create a node, labeled with the variable. For each clause, add an edge between the three nodes corresponding to the variables from that clause. Finally, for all i, add an edge between every pair of nodes labeled x_i and the other labeled $\neg x_i$.

The conversion of 2(c) is below:



3. (25 points)

- Let SAT be the following problem: Given a Boolean formula, in the following form:
$$C_1 \wedge \cdots \wedge C_L$$
where each C_i is a disjunction ($\vee$) of literals, is the formula satisfiable?
- Let k-SAT be the following problem: Given a Boolean formula, in the following form:
$$C_1 \wedge \cdots \wedge C_L$$
where each C_i is a disjunction ($\vee$) of *exactly* k literals, is the formula satisfiable?

The above two points are definitions, NOT questions. HERE are the questions:

(a) Show that 2-SAT is in P

(b) Show that $SAT \leq k\text{-}SAT$, for $k \geq 3$.

(PERMISSION: You may go to the web or elsewhere to find the answer; however, you must put it in your own words and understand your answer.)

SOLUTION TO PROBLEM THREE

- Note that any clause $(x_i \vee x_j)$ can be rewritten as the pair $(\neg x_i \implies x_j), (\neg x_j \implies x_i)$. Rewrite the original formula.
  Create the (directed) implication graph, where each variable and its negation is a node, and a directed edge exists for every implication. Then, an instance is satisfiable if and only if no literal and its negation belongs to the same strongly connected component of its implication graph.
- Consider the clause C_i. If it has only one literal, change it to $(L_1 \vee L_1 \vee L_1)$ If it has two literals, change it to $(L_1 \vee L_1 \vee L_2)$ If it has three literals, we are done. If it has more than three literals, then introduce new variables and replace the clause by:
$$(L_1 \vee L_2 \vee z_1) \wedge (\neg z_1 \vee L_3 \vee z_2) \wedge (\neg z_2 \vee L_4 \vee z_3) \wedge \dots$$