1. (5 points) What is your name? Write it clearly. Staple the HW.

2. (20 points) Let $A \in NP$. Let $p_1$, $p_2$ and $B$ be such that:

$$A = \{ x : (\exists y)[|y| = p_1(|x|) \land (x, y) \in B] \}$$

where $B$ is in $DTIME(p_2(n))$. $B$ is computed by a Turing machine that has 6 symbols in the alphabet and 10 states.

Let $x$ be a string of length $n$. Using the Cook-Levin Theorem, I can come up with a FORMULA $\phi$ such that $x \in A$ iff $\phi \in SAT$.

How many variables does $\phi$ have?

**SOLUTION TO PROBLEM TWO**

The Cook-Levin reduction assumes that the TM runs on the string $x\#y$ which is of length $n + p_1(n)$.

The computation runs in time $p_2(n + p_1(n))$.

Hence we have a table that is $(n + p_1(n))(p_2(n + p(1(n)))$.

For each entry of the table I have the variables $z_{i,j,\sigma}$ where $\sigma \in \Sigma \cup Q \times \Sigma$

There are $6 + (6 \times 10) = 66$ possible $\sigma$. Hence the number of variables is $66(n + p_1(n))(p_2(n + p(1(n))).$

**END OF SOLUTION TO PROBLEM TWO**

3. (20 points) Write down the part of the Cook-Levin Formula that corresponds to the transition:

$$\delta(q, a) = (p, b)$$

i.e. the head does not move and prints a ‘b’

**SOLUTION TO PROBLEM THREE**

$$z_{i,j,(q,a)} \Rightarrow z_{i,j+1,(p,b)}$$
KEY is that since the head does not move, nothing else changes.  

END OF SOLUTION TO PROBLEM THREE

4. (30 points) Show that if \( A \leq B \) and \( B \leq C \) then \( A \leq C \).

SOLUTION TO PROBLEM FOUR

Assume \( A \leq B \) via TM \( M_1 \) that runs in \( p_1(n) \) time, \( p_1 \) a poly.

Assume \( B \leq C \) via TM \( M_2 \) that runs in \( p_2(n) \) time, \( p_2 \) a poly.

The following is a reduction \( A \leq C \)

(a) Input(\( x \))

(b) Run \( M_1(x) \) to find \( y' \) (this takes \( p_1(|x|) \) steps. Note that \( |y'| \leq p(|x|) \)).

(c) Run \( M_2(y_1) \) to find \( y \) (this take \( p_2(|y'|) \leq p_2(p_1(|x|)) \)).

(d) Output \( y \).

\( x \in A \) iff \( y' \in B \) iff \( y \in C \).

The procedure takes \( p_2(p_1(|x|)) \) steps which is a polynomial.

END OF SOLUTION TO PROBLEM FOUR
5. (30 points) Let
\[ COL_2 = \{ G : G \text{ is 2-colorable} \} \]
\[ COL_3 = \{ G : G \text{ is 3-colorable} \} \]
\[ COL_4 = \{ G : G \text{ is 4-colorable} \} \]
\[ COL_5 = \{ G : G \text{ is 5-colorable} \} \]
\[ PCOL_2 = \{ G : G \text{ is Planar and 2-colorable} \} \]
\[ PCOL_3 = \{ G : G \text{ is Planar and 3-colorable} \} \]
\[ PCOL_4 = \{ G : G \text{ is Planar and 4-colorable} \} \]
\[ PCOL_5 = \{ G : G \text{ is Planar and 5-colorable} \} \]

For each statement below you must answer TRUE or FALSE and explain why. You may assume the following: (1) \( P \neq NP \), (2) \( COL_3 \) is NP-complete, (3) \( PCOL_3 \) is NP-complete, (4) \( SAT \) is NP-complete, (5) 3-SAT is NP-complete, (6) every planar graph is 4-colorable, (7) if \( A \leq B \) and \( B \leq C \) then \( A \leq C \).

As usual, \( A \leq B \) means that there is a function \( f \) computable in poly time such that, for all \( x \)

\[ x \in A \text{ iff } f(x) \in B \]

(a) \( COL_2 \leq COL_3 \)
(b) \( COL_3 \leq COL_2 \)
(c) \( COL_3 \leq COL_4 \)
(d) \( COL_4 \leq COL_3 \)
(e) \( PCOL_3 \leq COL_4 \)
(f) \( COL_3 \leq PCOL_4 \)

SOLUTION TO PROBLEM FIVE

(a) \( COL_2 \leq COL_3 \): TRUE. \( COL_2 \in P \). Anything in \( P \) is \( \leq \) anything.
(b) \( COL_3 \leq COL_2 \): FALSE. Since \( COL_2 \in P \), if this was true then \( COL_3 \in P \) but we are assuming \( P \neq NP \).
(c) $COL_3 \leq COL_4$: TRUE. The reduction is easy: given $G$, let $G'$ be $G$ with one more vertex added and connected to all vertices. $G$ is 3-colorable IFF $G'$ is 3-colorable.

(d) $COL_4 \leq COL_3$: TRUE. The reduction is INSANE: By Cook-Levin theorem $COL_4 \leq SAT$. By $COL_3$ being NP-complete, $SAT \leq COL_3$. Since reductions are transitive, $COL_4 \leq COL_3$.

(e) $COL_3 \leq PCOL_4$: FALSE. $PCOL_4$ is in P since ALL graphs are 4-colorable. If true this would imply $P = NP$.

END OF SOLUTION TO PROBLEM FIVE