HW 8 CMSC 452. Morally Due April 23
THIS HW IS TWO PAGES LONG!!!!!!!!!!

SOLUTIONS

1. (30 points) A *poly inequality* is an inequality of the form

\[ p(x_1, x_2, \ldots, x_n) \leq c \]

where \( p(x_1, \ldots, x_n) \) is a polynomial with integer coefficients WITHOUT a constant term, and \( c \in \mathbb{Z} \).

TWO EXAMPLES:

\[ x_1^3x_2^2 - 2x_2x_3 + 18x_3^4x_4^2 + x_1 \leq 1000. \]

\[ x_1 + x_2 \leq 89 \]

Let POLY PROGRAMMING, called \( PP \), be the following problem:

Given a set of poly inequalities determine if there is some way to set the variables to rationals so that all the inequalities hold.

(a) Show that 3-SAT \( \leq PP \).

(b) Use your reduction on the following formula (i.e., list the inequalities produced by the reduction)

\[ (x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor x_4) \land (x_2 \lor \neg x_3 \lor \neg x_4) \]

**GO TO THE NEXT PAGE!!!!!!!!!!!!**
(a) Given a formula $\phi(x_1, \ldots, x_n)$ output the following constraints

- For all $i$, $x_i^2 - x_i \leq 0$ and $x_i^2 - x_i \geq 0$. Hence we have $x_i^2 = x_i$ so $x_i = 0$ or $x_i = 1$.
- We now just use the same constraints based on clauses that we did for IP. For each clause $(L_1 \lor L_2 \lor L_3)$ do the following: If $L_i$ is a variable (not a negation of one) then let $M_i$ be that variable (e.g., $x_{18}$). If $L_i$ is a negation of a variable then let $M_i$ be $1 -$ that variable, (e.g., $1 - x_{18}$). Add the following linear inequality to the constraints:

$$M_1 + M_2 + M_3 \geq 1$$

(b) We first write the equations as they occur naturally, and then we put them in the right form.

$$x_1^2 - x_1 \leq 0$$
$$x_2^2 - x_1 \geq 0: \text{REWRITE} -x_1^2 + x_1 \leq 0$$
$$x_2^2 - x_2 \leq 0$$
$$x_2^2 - x_2 \geq 0: \text{REWRITE} -x_2^2 + x_2 \leq 0$$
$$x_3^2 - x_3 \leq 0$$
$$x_3^2 - x_3 \geq 0: \text{REWRITE} -x_3^2 + x_3 \leq 0$$
$$x_4^2 - x_4 \leq 0$$
$$x_4^2 - x_4 \geq 0: \text{REWRITE} -x_4^2 + x_4 \leq 0$$
$$x_1 + (1 - x_2) + x_3 \geq 1: \text{REWRITE} -x_1 + x_2 - x_3 \leq 0$$
$$(1 - x_1) + x_2 + x_4 \geq 1: \text{REWRITE} x_1 - x_2 - x_4 \leq 0$$
$$x_2 + (1 - x_3) + (1 - x_4) \geq 1: \text{REWRITE} -x_2 + x_3 + x_4 \leq 1$$

2. (40 points) Let

$$\text{CLIQ17} = \{G \mid \text{graph } G \text{ has a clique of size 17} \}$$

(a) (25 points) Either show that CLIQ17 is in P or show that CLIQ17 is NP-complete or do both. (ALSO — not to hand in, but think about — is it likely that someone in the class will be able to do both?)
(b) (25 points) Is CLIQ17 closed under minors (see Wikipedia entry for clarification). That is, if $G \in$ CLIQ17 and $H$ is a minor of $G$, is it necessarily true that $H \in$ CLIQ17? If so then prove it, if not then give a counterexample.

https://en.wikipedia.org/wiki/Graph_minor

**SOLUTION TO PROBLEM TWO**

a) CLIQ17 is in P:

- Input $G$
- For ALL sets of vertices of size 17 check if they form a clique. If any do then output YES, else NO.

This is in poly time since the number of sets of cliques to check is $\binom{n}{17} \leq n^{17}$.

THINK ABOUT PART: if someone proved that CLIQ17 was both in P and NP-complete then this would imply $P = NP$. This is unlikely to be true and unlikely to be proven by *anyone* at this time. But HEY — you never know!

b) CLIQ17 is NOT closed under minors. Take $K_{17}$. It is IN CLIQ17. Remove one vertex. Now it is $K_{16}$ which is NOT in CLIQ17.

3. (30 points) Let

$$FACT = \{(n, x) \mid \text{there is a nontrivial factor of } n \text{ that is } \leq x \}.$$  

(A NONTRIVIAL factor of $n$ is a positive factor that is NOT 1 and NOT $n$.)

$n$ and $x$ are both positive integers and are given in binary, so the NUMBER (say, for example) ONE THOUSAND only takes around 10 bits, NOT 1000 bits, to input.

Let $FFACT$ be the function that, on input $n$, outputs the complete prime factorization of $n$.

Show that if $FACT \in P$ then $FFACT$ can be computed in Polynomial time.
NOTE—poly in the LENGTH of the input. So the LENGTH of ONE THOUSAND would be TEN. So $FACT \in P$ means that it takes time $p(\log n + \log x)$ to decide $(n, x)$ for some poly $p$.

**SOLUTION TO PROBLEM THREE**

Note that if $n$ has a nontrivial factor then it has one $\leq \lceil \sqrt{n} \rceil$. We use this. If you used $n$ instead, that would be fine also.

Assume $FACT \in P$.

**ALGORITHM FOR FFACT.** We will be calling it recursively.

(a) Input $n$ (assume $n > 1$)

(b) Ask $(n, \lceil \sqrt{n} \rceil) \in FACT$? If NO then $n$ is prime so output $n$.

(c) If we got here then $(n, \lceil \sqrt{n} \rceil) \in FACT$. So we know there is a non-trivial factor of $n$ between 2 and $\lceil \sqrt{n} \rceil$. Do a binary search using queries to FFACT to find the LEAST such factor $m_1$. The number of queries is $O(\log \sqrt{n}) = O(\log n)$. Each one takes $O(p(\log n))$ steps. So $O(\log(n) \cdot p(\log n))$ time is taken.

(d) Note that $m_1$ is prime, and $m_2 = n/m_1$ is an integer.

(e) Call FFACT on $m_2$. Output $m_1$ in addition to the factorization of $m_2$ produced.

There will be only $O(\log n)$ recursive calls to FFACT. Hence, the total running time is $O((\log n)^2 \cdot p(\log n))$, which is polynomial in terms of $\log n$. 
