CMSC452 Final

1. You may have one piece of paper, 8.5 by 11, and a calculator.
2. There are 6 problems which add up to 100 points. (Problem 1 is *What is your name?* worth 0 points, so only 5 real problems.)
3. There are 20 pages. The last two pages are blank sheets for you to use for scratch work.
4. The exam is on Thursday May 15th, from 10:30AM-12:30PM unless you have contacted me to make other arrangements. So the exam is 2 hours.
5. Please write out the following statement: *I pledge on my honor that I will not give or receive any unauthorized assistance on this examination.*

1. (0 points but if you don't answer this one you may get a 0 on the exam) What is your name. PRINT CLEARLY

   DO NOT WRITE ON THE BACK OF ANY PAGE!!!!!!!!!!!!!!!!!!

2. (32 points) For this problem

   You will be asked to give examples of languages that satisfy some properties.

   NO JUSTIFICATION IS REQUIRED.

(a) (8 points) Give a language L such that the following hold:

- There is an NFA for L with ≤ 50 states.
- Every DFA for L requires ≥ 100 states.

NO JUSTIFICATION IS REQUIRED.

PUT ANSWER HERE:

ONE ANSWER:

$$\{a^n : n \neq 100\}$$

ANOTHER ANSWER:

$$\{a^n : n \equiv 0 \pmod{37}\} \vee \{a^n : n \equiv 0 \pmod{43}\}$$

GRADING: Some students used THE SAME MOD for both parts. THAT IS INCORRECT.

Some students did $\wedge$ instead of $\vee$. THAT IS INCORRECT.

(b) (8 points) Give a language L such that the following hold:

- $L \in \mathrm{NP}$.
- It is NOT known if L is in P.
- It is NOT known if L is NP-complete.

NO JUSTIFICATION IS REQUIRED.

PUT ANSWER HERE:

The following answers are correct:

- $\{(n,a) \mid (\exists b \leq a)[b$ divides n$\}$. We referred to this as FACT.
- $\{(G,H) \mid G$ and H are Isomorphic$\}$.
- Any complement of an NP-complete problem is also a correct answer.
- $\{n \mid (\exists x,y)[n = x^2 + y^2]\}$.

(c) (8 points) Give a language L such that the following hold:

- L is undecidable.
- The definition of L does not mention Turing Machines.

NO JUSTIFICATION IS REQUIRED.

PUT ANSWER HERE:

- The set of all polynomials $p \in \mathsf{Z}[x_1, \ldots, x_n]$ such that there exists $a_1, \ldots, a_n \in \mathsf{Z}$ with $p(a_1, \ldots, a_n) = 0$.
- The set of context free grammars G such that $\overline{L(G)}$ is also a context free grammar.

(d) (8 points) Give a language L such that the following hold:

- L is a set of graphs.
- L is NP-complete

(WARNING: Something like

$$\{(G, k) \colon G \text{ gets up and dances } k \text{ times }\}$$

is not allowed, since k is part of the input. The input has to JUST be a graph)

)

NO JUSTIFICATION IS REQUIRED.

PUT ANSWER HERE:

Recall that a graph $G = (V, E)$ is *3-colorable* if there exists $\mathrm{COL} \colon V \to \{1, 2, 3\}$ such that any two vertices connected by an edge have different colors.

$$\{G \colon G \text{ is 3-colorable}\}.$$

3. (16 points) In this problem we use the WS1S conventions.

   Give a DFA for

$$\{(x, y) : x \leq y\}.$$

   (All states are labelled A for accept or R for reject or S for stupid.)

   Use this page for Problem 3, and, if needed, the next page.

   I OMIT THE SOLUTION TO THIS ONE.

4. (16 points) For this problem we take the definition of Chomsky Normal Form a bit differently than usual.

   Every rule must be of one of the following forms

   $A \to BC$

   $A \to B$ (this is not usually allowed, but we allow it)

   $A \to \sigma$

   $S \to e$.

   For this problem $\Sigma = \{a, b\}$.

   (a) Give an algorithm that will do the following:

   Given $w_1, \ldots, w_n$ which are

   n* strings OF LENGTH *n,

   output a Chomsky Normal Form CFG G such that

   $$L(G) = \{w_1, \ldots, w_n\}$$

   (so G generates $\{w_1, \ldots, w_n\}$ and nothing else).

   Your algorithm can use DOT DOT DOT.

   (b) FILL IN the function f in the following statement:

   On input $\{w_1, \ldots, w_n\}$

   your algorithm produces a Chomsky Normal From CFG with $O(f(n))$ rules.

   Use the next one or two pages for your answer.

a) We describe the grammar.

Let $w_1 = \sigma_{11} \cdots \sigma_{1n}$

$\vdots \qquad \vdots \qquad \vdots$

Let $w_n = \sigma_{n1} \cdots \sigma_{nn}$

For $1 \le i \le n$ we have the following rules

$S_i \rightarrow [\sigma_{i1}][\sigma_{i2} \cdots \sigma_{in}]$

$[\sigma_{i2} \cdots \sigma_{in}] \rightarrow [\sigma_{i2}][\sigma_{i3} \cdots \sigma_{in}]$

$[\sigma_{i3} \cdots \sigma_{in}] \rightarrow [\sigma_{i3}][\sigma_{i4} \cdots \sigma_{in}]$

$\vdots \qquad \vdots \qquad \vdots$

$[\sigma_{i(n-1)}\sigma_{in}] \rightarrow [\sigma_{i(n-1)}][\sigma_{in}]$.

For each $1 \le i \le n$, $n-1$ rules. So $n(n-1)$ rules.

We need a few more rules:

$[a] \rightarrow a$

$[b] \rightarrow b$

$S \rightarrow S_1 \mid S_2 \mid \cdots \mid S_n$.

b) Number of rules is $n(n-1) + 2 + n = n^2 - n + 2 + n = n^2 + 2 = O(n^2)$.

GRADING: If your grammar uses much less than n^2 rules then it cannot work. I can prove it cannot work.

GRADING: Some students gave a CFG for the set of ALL strings of length n. This is not what we asked for.

5. (18 points) (You may do this problem on this page and, if needed, the next page.)

   For this problem if f is a function from N to N then

   IMAGE(f), the image of f, is

$$\{y\colon (\exists x \in \mathsf{N})[f(x) = y]\}.$$

   So this is the set of values of N that some element of f maps to.

   Prove the following:

   Let f be a computable function such that, for all x, y

   $x < y$ implies $f(x) < f(y)$.

   (so f is strictly increasing).

   Then IMAGE(f) *is DECIDABLE.*

   Use the next one or two pages for your answer.

Page for Problem 5.

(a) Input y

(b) For $x = 1$ to infinity do the following

(c) i. If $f(x) < y$ then go to the next iteration of the for loop.

   ii. If $f(x) = y$ then output YES and STOP

   iii. If $f(x) > y$ then output NOT and STOP (It must be the case that $f(x-1) < y$ and $f(x) > y$. Since f is strictly increasing we KNOW that $f(x)$ will NEVER be y.

6. (18 points)

   Let

$$\mathrm{COL}_3 = \{G\colon G \text{ is 3-colorable}\}$$

$$\mathrm{COL}_4 = \{G\colon G \text{ is 4-colorable}\}$$

   Show that $\mathrm{COL}_3 \le \mathrm{COL}_4$.

   (You will need to describe a function f takes as input a graph G and outputs a graph G' such that

   G is 3-colorable IFF G' is 4-colorable.)

   Use the next one or two pages for your answer.

SOLUTION

(a) Input $G = (V, E)$.

(b) We need to create G'.

(c) Let w be a NEW vertex.

(d) **Intuition** G' is G with w added and every vertex in G has an edge to w.

Formally $G' = (V', E')$ where

$V' = V \cup \{w\}$.

$E' = E \cup \{(w, v) \colon v \in V\}$

END OF SOLUTION