BILL AND NATHAN, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

Factoring
Is Probably Not NPC

BILL START RECORDING

Factoring: Some History

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I think it is unlikely that anyone aside from myself will ever know.

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We can now factor J easily. Was Jevons' comment stupid?

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  We know about the role of computers to speed up calculations, but it's reasonable it never dawned on him.
- **Conclusion**
  - His arrogance: assumed the world would not change much.
  - Our arrogance: knowing how much the world did change.

Factoring Algorithms

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- SVP algorithm (2020): Unclear!

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This is an informal diff between Factoring and SAT.

Factoring as a Set

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So our questions is: is FACT NPC?

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Note $\mathrm{3SAT}$, HAM, EUL, CLIQ are all in NP.

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We do not think $\mathrm{TAUT} \in \mathrm{NP}$.

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2. Number Theorists think $\text{FACT} \in \text{P}$.
3. Quantum Computing People thing quantum computers will factor very large numbers within 30 years. They are wrong.

Primality in NP

What we Know about Primality

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The next X slides have the proof that PRIMES is in NP. We omit them in our talk and you will not be responsible for it.
We need PRIMES in NP for our proof that $\overline{\text{FACT}} \in \text{NP}$.

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We abbreviate **certificate** by **cert**.

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2. for every factor $q \neq 1$ of $n-1$, $a^{(n-1)/q} \not\equiv 1 \pmod{n}$,

then n is prime.

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2. A factorization of $n-1 = p_1^{c_1} \cdots p_k^{c_k}$ where p_i's are prime.

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So need the cert to contain a cert that the claimed prime factors of $n-1$ are prime.

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So it's a recursive cert.
Need to check that the cert is short, but this is not difficult.

Back to Factoring

$\overline{\textbf{FACT}} \in \textbf{NP}$

$$\mathrm{FACT} = \{(n, a) : (\exists b \leq a)[b \text{ divides } n]\}$$

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Verifier has to check

1. $n = p_1^{c_1} \cdots p_k^{c_k}$.
2. $a < p_1$.
3. Each p_i is prime.

Recap What We Know

We know $\overline{\mathrm{FACT}} \in \mathrm{NP}$

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Hence we do not think FACT is NPC.

So thats what I mean by **Factoring is probably not NPC**.

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  - Nolawe, Isaac, and Felix have not worked on it yet.

BILL AND NATHAN STOP RECORDING