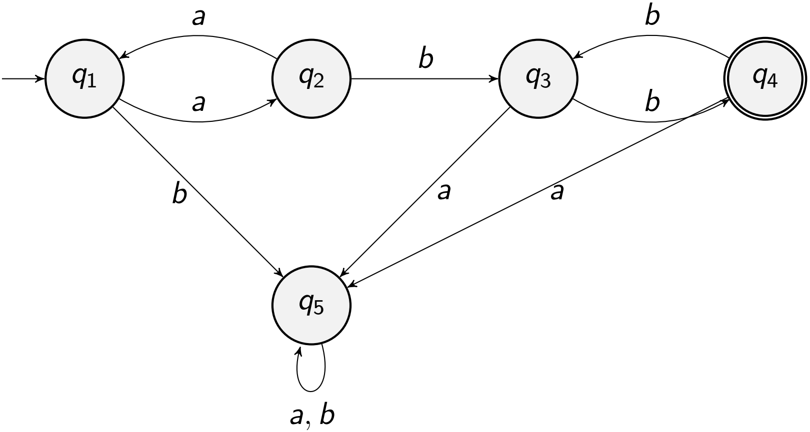
Review for CMSC 452 Midterm

Deterministic Finite Automata (DFA)

DFA Diagram

DFA Diagram



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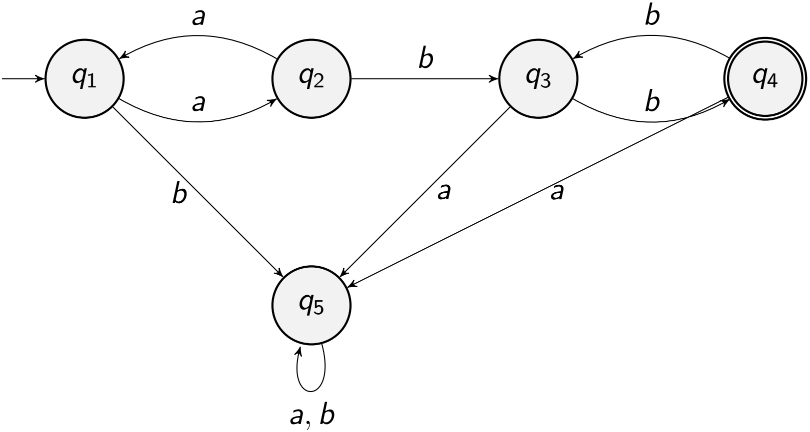
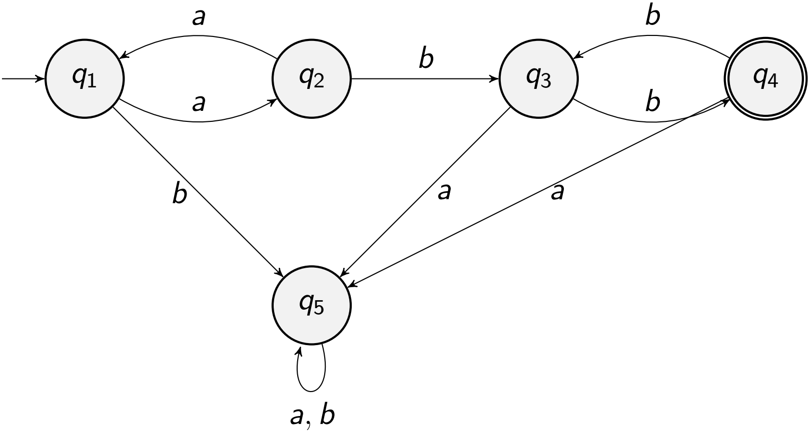


What is the language?

DFA Diagram



What is the language?

Odd number of a's followed by an even number of b's, but at least two.

$$\{w : \#_a(w) \equiv 1 \pmod 2 \wedge \#_b(w) \equiv 2 \pmod 3\}$$

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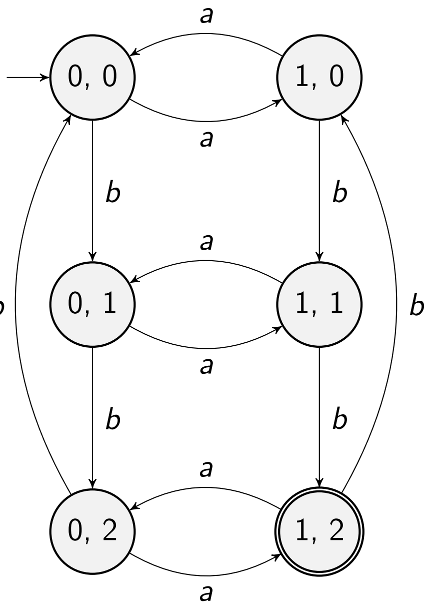

Transition Table

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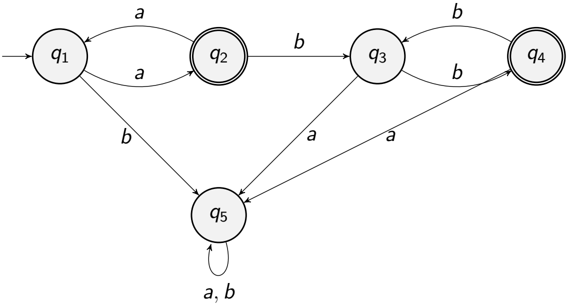

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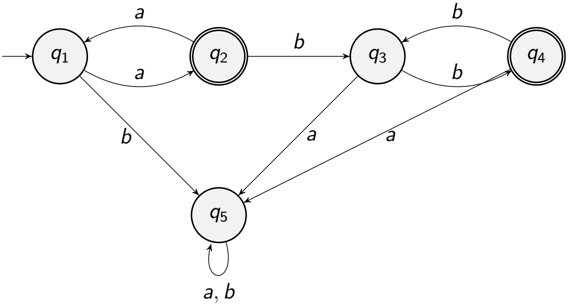

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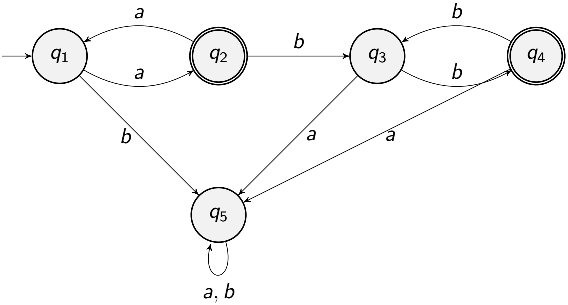

Transition Table:

- States: $\{q_1, q_2, q_3, q_4, q_5\}$

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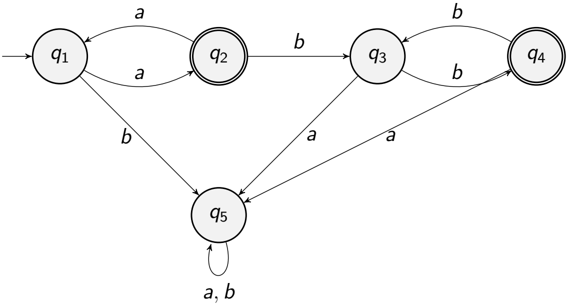

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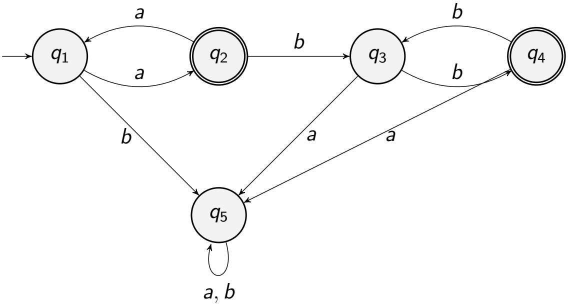

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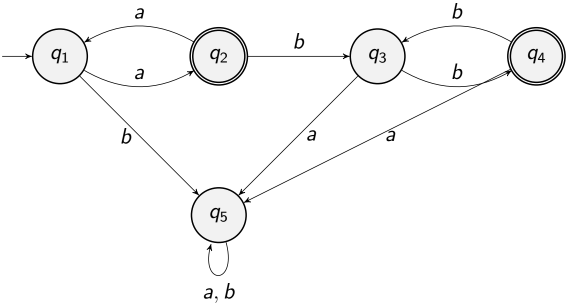

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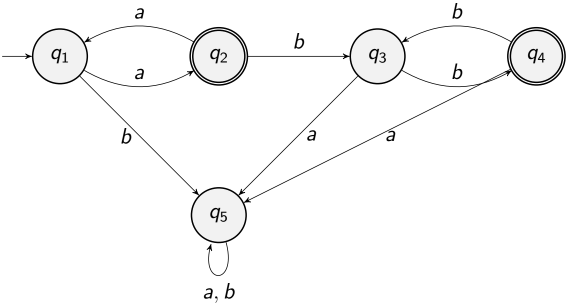

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- Transition function

	a	b
q_1	q_2	q_5
q_2	q_1	q_3
q_3	q_5	q_4
q_4	q_5	q_3
q_5	q_5	q_5

Divisibility

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We get a DFA (a trick?) for Mod 11.

Trick for Mod 11. All $\equiv$ are Mod 11

Is there a trick for mod 11?

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Pattern is $1, -1, 1, -1, \ldots$.

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Thm $d_n \cdots d_0 \equiv d_0 - d_1 + d_2 - \cdots \pm d_n.$

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Thm $d_n \cdots d_0 \equiv d_0 - d_1 + d_2 - \cdots \pm d_n$.

Proof may be on HW or Midterm or Final or some combination.

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Need to keep track of both the running weighted sum mod 11 and if you are reading an even or odd place.

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$$\delta((i,j),\sigma) \begin{cases} (i+\sigma \pmod{11}, j+1 \pmod{2}) \text{ if } j=0 \\ (i-\sigma \pmod{11}, j+1 \pmod{2}) \text{ if } j=1 \end{cases} \tag{1}$$

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Classifier If end in $(i, 0)$ or $(i, 1)$ then number is $\equiv i$.

Mods Can Get More Complicated: Mod 224

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Nondeterministic Finite Automata (NFA)

NFA's Intuitively

1. An NFA is a DFA that can guess.
2. NFAs do not really exist.
3. Good for $\cup$ since can guess which one.
4. An NFA accepts iff SOME guess accepts.

Every NFA-lang a DFA-lang!

Thm If L is accepted by an NFA then L is accepted by a DFA.
Pf Sketch L is accepted by NFA $(Q, \Sigma, \Delta, s, F)$ where

1. Get rid of e-transitions using reachability.
2. Get rid of non-determinism by using power sets. Possibly 2^n blowup.

Regular Expressions

Examples

1. $b^*(ab^*ab^*)^*ab^*$
2. $b^*(ab^*ab^*ab^*)^*$
3. $(b^*(ab^*ab^*)^*ab^*) \cup (b^*(ab^*ab^*ab^*)^*)$

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Lemma If a language is generated by a regular expression, it is recognized by an NFA.

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Base Cases $|\alpha| = 1$. Then $\alpha = e$ or $\alpha = \sigma$.

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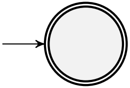

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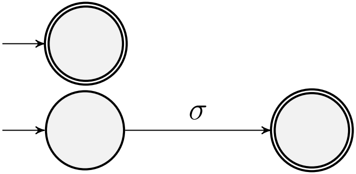

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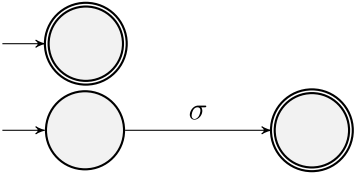

We skip rest of the proof.

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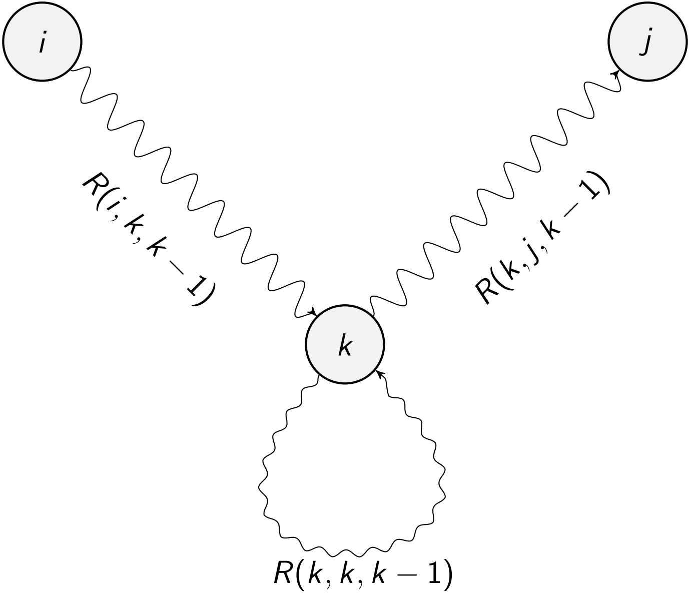
$R(i, j, k) = \{w : \delta(i, w) = j$ but only use states in $\{1, \ldots, k\}$ $\}$.

Inductive Step $R(i, j, k)$ as a Picture



Complete Proof on One Slide

For all $1 \leq i, j \leq n$:

$$R(i,j,0) = \begin{cases} \{\sigma : \delta(i,\sigma) = j\} & \text{if } i \neq j \ \} \\ \{\sigma : \delta(i,\sigma) = j\} \cup \{e\} & \text{if } i = j \ \} \end{cases} \qquad (2)$$

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All $R(i,j,0)$ are Regex.

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All $R(i,j,0)$ are Regex.

For all $1 \leq i, j \leq n$ and all $0 \leq k \leq n$

$$R(i,j,k) = R(i,j,k-1) \bigcup R(i,k,k-1) R(k,k,k-1)^* R(k,j,k-1)$$

Complete Proof on One Slide

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If ALL $R(i,j,k-1)$ are Regex, then ALL $R(i,j,k)$ are Regex.

Textbook Regular Expressions

We allow numbers as exponents. For example the following is not a regex but is a trex:

$$\{a, b\}^* a \{a, b\}^n.$$

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Often the trex is shorter than the regex.

Closure Properties

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Property	DFA	NFA	regex
$L_1 \cup L_2$	Prod	e-trans	Def
$L_1 \cap L_2$	Prod	Prod	X
$\overline{L}$	Swap	X	X
$L_1 \cdot L_2$	X	e-trans	Def
L^*	X	e-trans	Def

Summary of Blowup for Closure Properties

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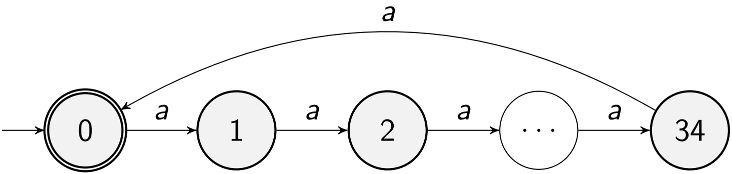
Closure Property	DFA	NFA	Regex
$L_1 \cup L_2$	$n_1 n_2$	$n_1 + n_2 + 1$	$\ell_1 + \ell_2$
$L_1 \cap L_2$	$n_1 n_2$	$n_1 n_2$	X
$L_1 \cdot L_2$	X	$n_1 + n_2$	$\ell_1 + \ell_2$
$\overline{L}$	n	X	X
L^*	X	$n + 1$	$\ell + 1$

Number of States for DFAs and NFAs

Minimal DFA for $L_1 = \{a^i : i \equiv 0 \pmod{35}\}$



Min DFA for $L_2 = \{a^i : i \not\equiv 0 \pmod{35}\}$

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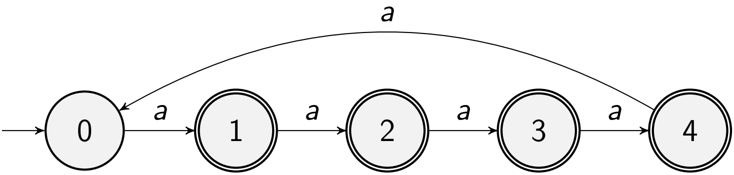
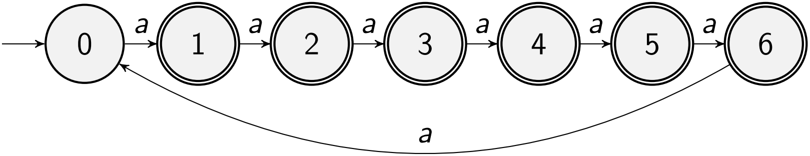
$\exists$ DFA for L_2: 35 states: swap final-$\overline{\text{final}}$ states in DFA for L_1.

Small NFA for $L_2 = \{a^i : i \not\equiv 0 \pmod{35}\}$

Need these two NFA's.





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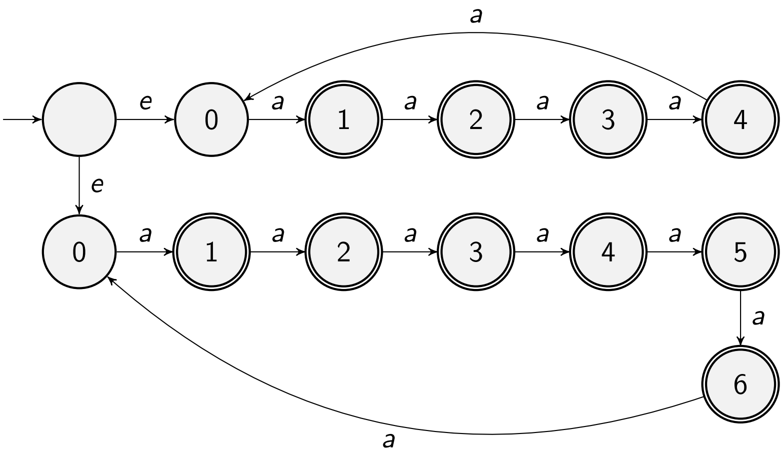

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NFA for L_2 can be done with $1 + 5 + 7 = 13$ states.

DFA for $L_4 = \{a^i : i \neq 1000\}$

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1. There is a DFA for L_4 that has 1000 states.
2. Any DFA for L_3 has ≥ 1000 states.

Small NFA for $L_4 = \{a^n : n \neq 1000\}$

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Create the union of NFA's A and B.

Sums of 32's and 33's

Thm

1) $(\forall n \geq 1001)(\exists x, y \in \mathbb{N})[n = 32x + 33y + 9]$.

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2) $(\neg \exists x, y \in \mathbb{N})[1000 = 32x + 33y + 9]$.

NFA A

Idea Start state, then 8 states, then a loop of size 33 with a shortcut at 32.

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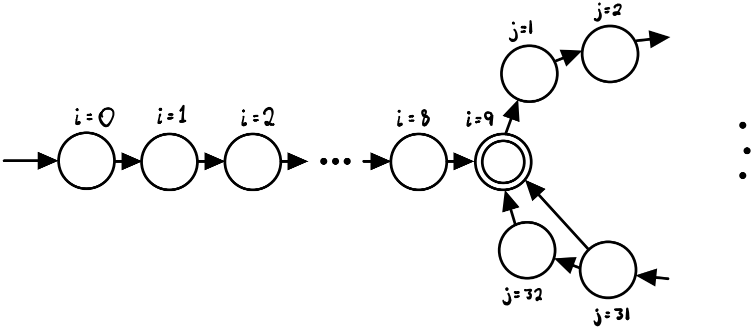

Why Works for $\{a^i : i \geq 1001\}$ and More

By the loop Theorem for 32, 33, the NFA

1. Accepts $\boldsymbol{\{a^i : i \geq 1001\}}$.
2. Might accept more.
3. DOES NOT accept a^{1000}.

Number of States for $\{a^i : i \geq 1001\}$

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Total number of states: $9 + 33 = 42$.

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$1000 \equiv 10 \pmod{11}$ 11-state DFA for $\{a^i : i \not\equiv 10 \pmod{11}\}$.

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Could go on to 13,17, etc. But we will see we can stop here.

Machine B

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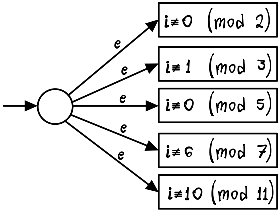

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

Thm Let M be the NFA from the last slide with the Mods.
$M(a^{1000})$ is rejected. This is obvious.

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We omit the proof that it works but note that we use that the product of the mods

$$2 \times 3 \times 5 \times 7 \times 11 = 2310 > 1000.$$

How Many States for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}?

$2 + 3 + 5 + 7 + 11 = 28$ states.
Plus the start state, so 29.

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Take NFA of union using e-transitions for an NFA and do not count start state twice, so have

$$42 + 29 - 1 = 70 \text{ states.}$$

Can We Do Better than 70 States?

YES–59 states:

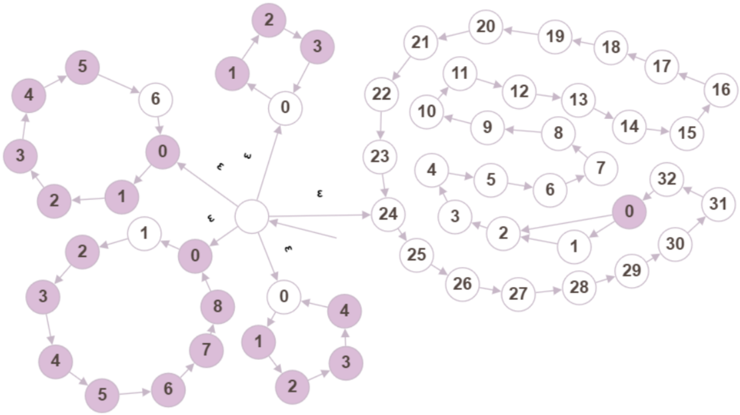

Figure: 59 State NFA for L_4

Math Needed for $\{a^i : i \neq n\}$

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$xy - x - y + t = n + 1$.
This leads to loops and tail that are roughly $\leq 2\sqrt{n}$ states.

Proving That a Language Is Not Regular

Pumping Lemma (PL)

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Proof

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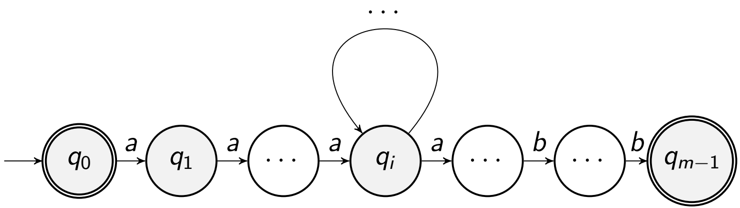

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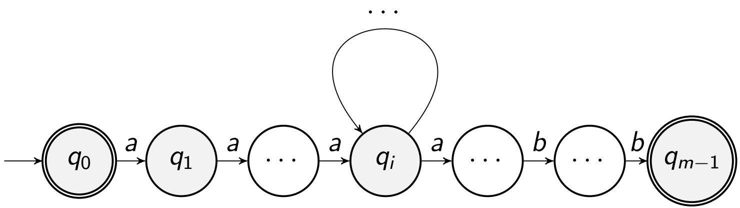

$a^{n+k} b^n$ is accepted by following the loop again. Contradiction.

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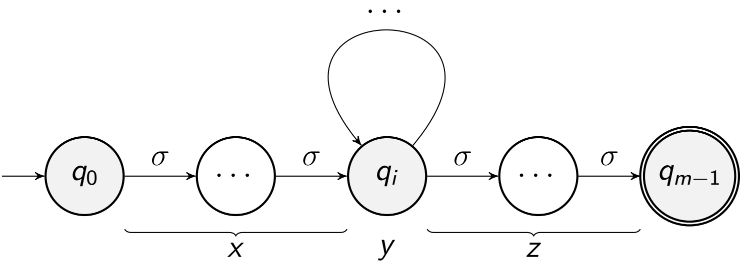

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We then find some i such that $xy^iz \notin L$ for the contradiction.

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Contradiction since $k \geq 1$.

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If L_3 is regular then $L_2 = \overline{L_3}$ is regular. But we know that L_2 is not regular. DONE!

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We Omit the Formal Proof.