BILL AND NATHAN RECORD LECTURE!!!!

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Nondeterministic Finite Automata (NFA): Closure Properties

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We will keep track of number-of-states.

Reg Langs Closed Under Complementation

How do you complement a reg lang (not a joke)?

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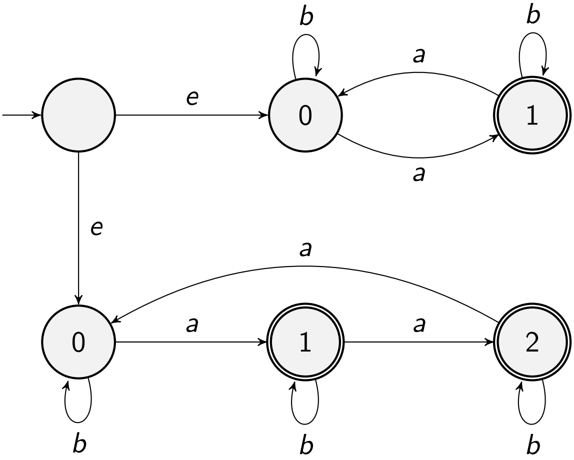
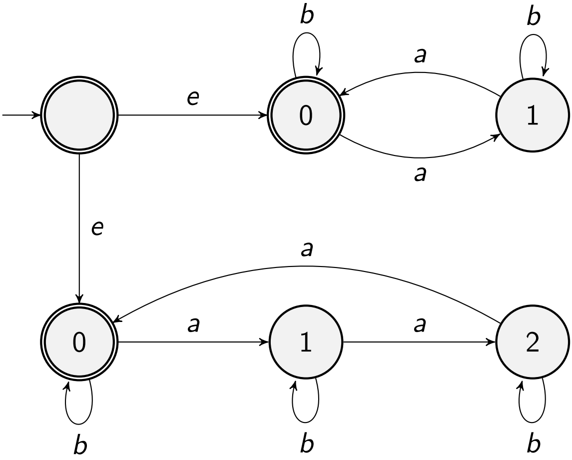
See next slide.

$\{a^n : n \not\equiv 0 \pmod 6\}$



Final and Non-final States Swapped



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No. There are langs L where:

- there is an NFA for L is size n.
- any NFA for $\overline{L}$ is of size $\sim 2^n$.
  See next slide for this example.

Example of a Blowup for Complementation

Example of a language L_n such that

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2. Any NFA for $\overline{L_n}$ requires size $\Omega(p_1 p_2 \cdots p_n) = \Omega(e^{n \log n})$.

Reg Langs Closed Under Union-Intuition

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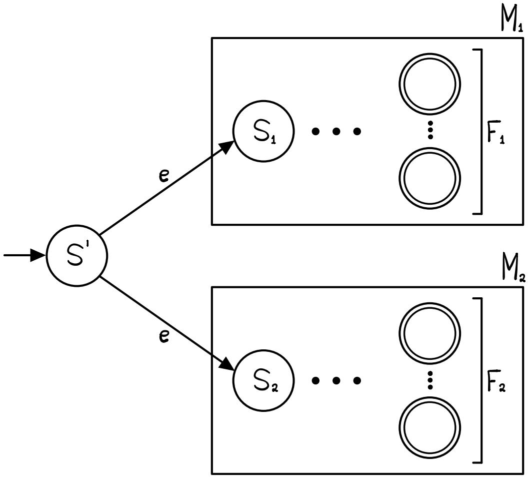

Reg Langs Closed Under Union-Formally

Formally If L_1 is reg via NFA

$$(Q_1, \Sigma, \Delta_1, s_1, F_1). \text{ We will take } |Q_1| = n_1 \ .$$

and L_2 is reg via NFA

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NOT a win: We get small NFA, not small DFA.

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Answer Option 2: Can do with NFAs but gets $n_1 n_2$ states.
It is a cross product construction. Next Slide.

Reg Langs Closed Under Intersection: Proof

Let $M_1 = (Q_1, \Sigma, \Delta_1, s_1, F_1)$ be an NFA for L_1
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From M_1 and M_2 construct an NFA M for $L_1 \cap L_2$.

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Reg Langs Closed Under Concat-Intuitively

Have an e-transition from final state of M_1 to start state of M_2.

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Generic picture on next slide.

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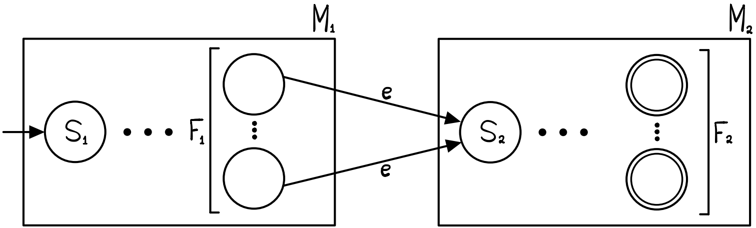

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Number of states: $n_1 + n_2$.

Reg Langs Closed Under $*$?-READ ON YOUR OWN

The next few slides are on closure under $*$.

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We skip to the end.

Reg Langs Closed Under $*$?-Intuition-1st Try

Have an e-transition from final states of M to start state of M.

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Next slide has a generic picture of this approach.

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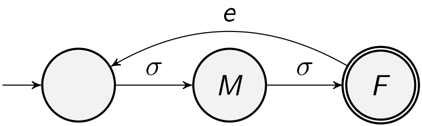
Spoiler Alert This will not work.

Reg Langs Closed Under ∗?-Picture-1st Try



What Goes Wrong with 1st Try?

What goes wrong?

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We want e to be accepted.

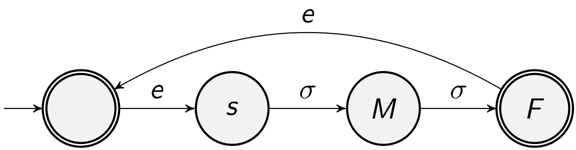
What Goes Wrong with 1st Try?

What goes wrong?

We want e to be accepted.

Next slide has an NFA tht does work.

Reg Langs Closed Under ∗?-Picture-3rd Try



Reg Langs Closed Under $*$?-Formally

Might be a HW or exam question.

Summary of Closure Properties and Proofs

X means **can't prove easily**
$n_1 + n_2$ (and similar) is number of states in new machine if L_i reg via n_i-state machine.

Closure Property	DFA	NFA
$L_1 \cup L_2$	$n_1 n_2$	$n_1 + n_2 + 1$
$L_1 \cap L_2$	$n_1 n_2$	$n_1 n_2$
$L_1 \cdot L_2$	X	$n_1 + n_2$
$\overline{L}$	n	X
L^*	X	$n + 1$

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