Proving That a Language Is Not Regular

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Three ways to represent regular languages (so far)

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- **Method 2—Pumping Lemma:** Run the DFA on one long word. By the **PHP** the word must visit the same state twice. Then do some **magic**.

Method 1

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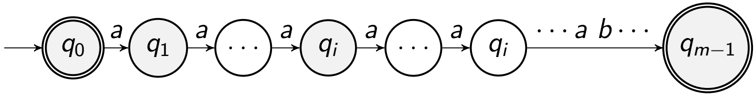

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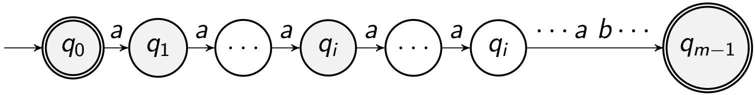

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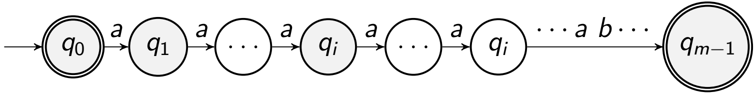

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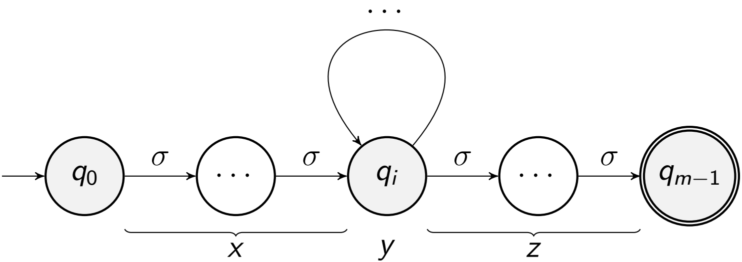

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We then find some i such that $xy^iz \notin L$ for the contradiction.

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$$a^{n+k} b^n \in L_1$$

REDO: $L_1 = \{a^n b^n : n \in \mathbb{N}\}$ is Not Regular

Assume L_1 is regular.
By Pumping Lemma, for long $a^n b^n \in L_1$, $\exists x, y, z$:

1. $y \neq e$.
2. $|xy|$ is short.
3. For all $i \geq 0$, $xy^i z \in L_1$.

Take w long enough so that the xy part only has a's.
$x = a^j$, $y = a^k$, $z = a^{n-j-k} b^n$. Note $k \geq 1$.
By the Pumping Lemma, all of the words

$$a^j \left(a^k\right)^i a^{n-j-k} b^n \;=\; a^{n+k(i-1)} b^n$$

are in L_1.
Take $i = 2$ to get

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Contradiction since $k \geq 1$.

$L_2 = \{w : \#_a(w) = \#_b(w)\}$ is Not Regular

Proof: Same Proof as L_1 not Reg: Still look at $a^m b^m$.
Key Pumping Lemma says for ALL long enough $w \in L$.

$L_3 = \{w : \#_a(w) \neq \#_b(w)\}$ is Not Regular

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Pumping Lemma Does Not Help. When you increase the number of y's there is no way to control it so carefully to make the number of a's EQUAL the number of b's.

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So what do to?
Think some more.

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If L_3 is regular then $L_2 = \overline{L_3}$ is regular. But we know that L_2 is not regular. DONE!

$L_4 = \{a^{n^2} : n \in \mathbb{N}\}$ is Not Regular

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By Pumping Lemma for long enough $a^{n^2} \in L_4$ there exist $x = a^j$, $y = a^k$ $(k \neq 0)$, $z = a^\ell$ with $xyz = a^{n^2}$. Also

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$(\forall i \geq 1)[j + k + \ell + (i-1)k$ is a square$]$.

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See slide for exciting finish!

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$(\forall i \ge 1)[n^2 + (i-1)k$ is a square].
So $n^2, n^2 + k, n^2 + 2k, \ldots$ are all squares.

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$n^2 + 2k \geq (n+2)^2 = n^2 + 4n + 4$. So $k \geq 2n + 2$.

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So
$(\forall i \ge 1)[n^2+ik \ge (n+i)^2 = n^2+2in+i^2]$.

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So
$(\forall i \geq 1)[n^2 + ik \geq (n+i)^2 = n^2 + 2in + i^2]$. So
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So k is bigger than any natural number!

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So k is bigger than any natural number!
Contradiction.

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By Pumping Lemma, for large p, $a^p \in L_5$ $\exists$ $x = a^j$, $y = a^k$, $z = a^\ell$ such that

$$a^j (a^k)^i a^\ell \in L_5$$

$$(\forall i \geq 0)[j + ik + \ell \text{ is prime}].$$

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Discuss: can we get a contradiction out of this?

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Let $i = j + \ell$.
$j + (j + \ell)k + \ell$ is prime

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$$(\forall i \geq 0)[j + ik + \ell \text{ is prime}].$$

Let $i = j + \ell$.
$j + (j + \ell)k + \ell$ is prime
$(j + \ell + (j + \ell)k$ is prime.
$(j + \ell)(1 + k)$ is prime.
Contradiction.
(There are some cases to work out like what if $(j + \ell) = 1$ but we skip this part.)

$L_6 = \{\#_a(w) > \#_b(w)\}$ is Not Regular

We will be brief here.

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Take $w = b^n a^{n+1}$, long enough so the y-part is in the b's.

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We will be brief here.
Take $w = b^n a^{n+1}$, long enough so the y-part is in the b's.
Pump the y to get more b's than a's.

$L_7 = \{a^n b^m : n > m\}$ is Not Regular

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Think about.

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Can **also** bound $|yz|$ by same proof.

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Do that and then you can get y to be all b's, pump b's, and get out of the language.

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$$xy^i z = a^{j+ik+(n-j-k)} b^{n-1} c^n$$

$L_8 = \{a^{n_1} b^m c^{n_2} : n_1, n_2 > m\}$ is Not Regular (Cont)

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Key We are used to thinking of i large.

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$$xy^0 z = a^{n-k}b^{n-1}c^n$$

Since $k \geq 1$, we have that $\#_a(xy^0z) < n \leq n-1 = \#_b(xy^0z)$.
Hence $xy^0z \notin L_8$.

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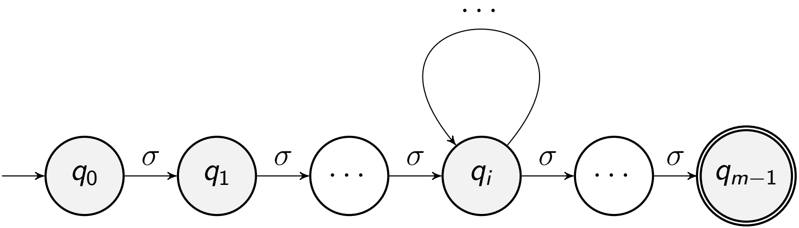
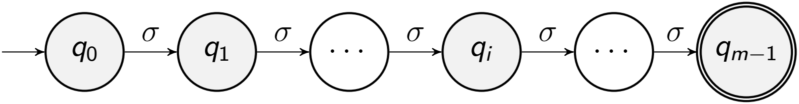
Hence $xy^0 z \notin L_8$. Contradiction.

$i = 0$ Case as a Picture





Lower Bounds: Looking Ahead

1. DFA's are simple enough devices that we can actually prove languages are not regular
2. We will later see that Context Free Grammars are simple enough devices that we can prove Languages are not Context Free.
3. Poly-bounded Turing Machines seem to be complicated devices, so proving P$\neq$NP seems to be hard.

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1. DFA's are simple enough devices that we can actually prove languages are not regular
2. We will later see that Context Free Grammars are simple enough devices that we can prove Languages are not Context Free.
3. Poly-bounded Turing Machines seem to be complicated devices, so proving P$\neq$NP seems to be hard.
   However, I expect the TA work it out by the end of the semester.
4. Proving problems undecidable is surprisingly easy since such proofs do not depend on the details of the model of computation.