Quantum Bits, Entanglement, and the CHSH Game

Exposition by
William Gasarch and Evan Golub

May 2, 2025

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5. We STATE that there is a strategy for the CHSH game where (1) the 2 players have qubits that are entangled, and (2) **the prob of winning is larger than 0.75.**
6. We discuss what this all means.

Quantum Bits I: Measure Once

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This matrix rotates vectors by θ.

Quantum Bits

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We will elaborate on this on the next slide.

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She will get 0 with prob γ^2 and 1 with prob δ^2
So she changes the state of the qubit before measuring it.
This is referred to as **measuring the qubit in a different basis** or in a **different frame.**

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Next two slides have the first and second coordinate of $M_{\frac{\pi}{6}}(v)$

Example (cont)

First coordinate of $M_{\frac{\pi}{6}}(v)$ is
$\cos(\theta)\alpha - \sin(\theta)\beta = \cos(\frac{\pi}{6})\frac{1}{\sqrt{2}} - \sin(\frac{\pi}{6})\frac{1}{\sqrt{2}}$

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$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} - \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

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as $0 \le \theta \le \frac{\pi}{4}$, $\Pr(0)$ goes from $\frac{1}{2}$ to 0.

Quantum Bits II: Measure Twice

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Measuring a Qubit Twice In Same Basis

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Scenario 1:

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Upshot If use same basis then they will **agree**.

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The prob that Bob gets b is $\cos^2(\theta_1 - \theta_2)$.

The CHSH Game

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   If $x \wedge y = 0$ then Alice and Bob win if $a = b$.
   If $x \wedge y = 1$ then Alice and Bob win if $a \neq b$.

Classic Strategies

On the next few slides we discuss strategies with an eye towards asking how often they win.

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0	1	0	0	0	Y	Y
1	0	0	0	0	Y	Y
1	1	0	0	1	Y	N

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0	1	0	0	0	Y	Y
1	0	0	0	0	Y	Y
1	1	0	0	1	Y	N

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If Alice and Bob share an EPR pair then have a strategy that wins the CHSH game with probability $\cos^2(\pi/8) \sim 0.854 > 0.75$.

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4. This fits into our framework of upper and lower bounds on problems in different models.

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3. The question of if Alice and Bob can do better than $\cos^2(\frac{\pi}{8})$ is **Unknown to Science**.

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Comments on the answer on the next page.

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I am glad I am in math where we have well defined problems.

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1. Classically: There is a strategy for CHSH that has prob of winning 0.75 and it is known you cannot do better than that.
2. If Alice and Bob share an EPR pair then there is a strategy that has prob of winning $\cos^2(\frac{\pi}{8}) \sim 0.853$.
3. I am amazed that with a shared EPR pair Alice and Bob can do better.
4. I am amazed that with a shared EPR pair Alice and Bob can do **so much better**. I would have have thought something like $0.75 + \epsilon$.
5. Even with many EPR pairs and any kind of strategy Alice and Bob cannot do better than $\cos^2(\frac{\pi}{8})$. I am not amazed this is true, but I am amazed its been proven. (Proof is hard.)