Cryptography

Lecture 08

Pseudorandom Functions and Permutations

Keyed functions

- Let $F : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^n$ be an efficient, deterministic algorithm
  - Define $F_k(x) = F(k, x)$
  - The first input is called the key
- Choosing a uniform $k \in \{0,1\}^n$ is equivalent to choosing the function $F_k : \{0,1\}^n \rightarrow \{0,1\}^n$
  - i.e. for fixed key length n, the algorithm F defines a distribution over functions in $Func_n$!

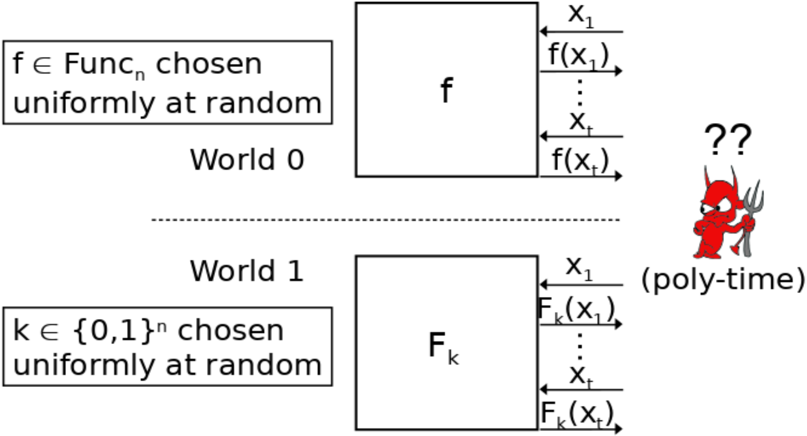
Note: A Keyed Perm requires F_k a perm and F_k^{-1} easy to compute.

Pseudorandom Functions (PRFs)

We define Pseudorandom Function informally.

A Pseudorandom Function is a keyed function $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$ such that a PPT Eve cannot do well in the following game:

1. Alice picks $k \in \{0,1\}^n$ and hence picks F_k
2. Bob picks a function f uniformly at random from $func_n$.
3. Eve gets a black box for one of $\{F_k, f\}$.
4. Eve needs to determine which one.



f ∈ $Func_n$ chosen uniformly at random
World 0
f
x_1
$f(x_1)$
x_t
$f(x_t)$
??
(poly-time)
World 1
k ∈ $\{0,1\}^n$ chosen uniformly at random
F_k
x_1
$F_k(x_1)$
x_t
$F_k(x_t)$

Pseudorandom Permutations (PRPs)

We define Pseudorandom Permutation informally.

A Pseudorandom Permutation is a keyed function $F : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^n$ such that every F_k is a permutation and a PPT Eve cannot do well in the following game:

1. Alice picks $k \in \{0,1\}^n$ and hence picks F_k
2. Bob picks a permutation f uniformly from $perm_n$.
3. Eve gets a black box for one of $\{F_k, f\}$.
4. Eve needs to determine which one.

Note:

- For large enough n, a random permutation is indistinguishable from a random function
- So in Psuedorandom Function game Bob could pick a random permutation.

PRFunctions Yields PRGenerators

- PRF F immediately implies a PRG G:
  - Define $G(k) = F_k(0 \cdots 0) \mid F_k(0 \cdots 1) \mid \cdots F_k(1 \cdots 1)$
- PRF can be viewed as a PRG with random access to exponentially long output
  - The function F_k can be viewed as the $n2^n$-bit string $F_k(0 \ldots 0) \mid \cdots \mid F_K(1 \ldots 1)$

Do PRFs/PRPs exist? Theoretical Answer

A one-way function (perm) is function (perm): easy to compute, hard to invert.
A one-way function (perm) with a hard core predicate is a function (perm) that is easy to compute but hard to invert, and (say) the middle bit of $f^{-1}(x)$ is hard to compute.

Chapter 7 shows:
$\exists$ One way Perm $\implies$ $\exists$ one way perm with a hcp.
$\exists$ one way perm with hcp $\implies$ $\exists$ PRG with expanion 1
$\exists$ PRG with expa-1 $\implies$ $\exists$ PRG with expa-$p(n)$ any poly p.
$\exists$ PRG with expa-$2n$ $\implies$ $\exists$ PRF.
Note: One way func $\implies$ PRF also known but much harder.

Comment on Theoretical Answer

Could start with a function that we thing is a One Way Perm.
Can you think of one? Discuss

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If p is a prime and g is a generator than $f(x) = g^x \pmod{p}$:

1. f is a perm.
2. If we think Discrete Log is hard then f is not invertible.

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DL hard $\implies$ f is one-way-perm $\implies \cdots \implies$ PRF.
Should we construct one this way?Discuss

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Should we construct one this way?Discuss
No: Too slow. But good for proof of concept.

Do PRFs/PRPs exist? Practical

- Block ciphers are practical constructions of pseudorandom permutations
- No asymptotics: $F : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}^m$
  - n = "key length"
  - m = "block length"
- Hard to distinguish F_k from uniform $f \in Perm_m$ *even for attackers running in time* $\approx 2^n$

AES

- Advanced encryption standard (AES)
  - Standardized by NIST in 2000 based on a public, worldwide competition lasting over 3 years
  - Block length = 128 bits
  - Key length = 128, 192, or 256 bits
- Will discuss details later in the course
- Currently no reason to use anything else

Recall Comp CPA-security via a Game.

Π is an encryption system. n is a security param.

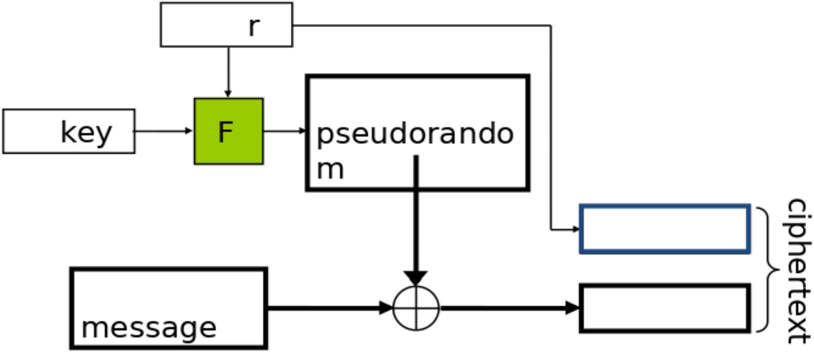
1. $k \leftarrow Gen(1^n)$. Eve does NOT know k.
2. Eve picks $m_0, m_1 \in \mathcal{M}$ $(|m_0| = |m_1|)$. Eve has BB for Enc_k.
3. $b \leftarrow \{0,1\}$, $c \leftarrow Enc_k(m_b)$
4. Π sends c to Eve.
5. Eve outputs $b\text{`} \in \{0,1\}$. Eve has BB for Enc_k.
6. If $b = b'$ then Eve *Wins!*

Π Comp CPA-secure if for all PPT Eve

$$\Pr[\text{Eve Wins}] \le \frac{1}{2} + \varepsilon(n)$$

CPA-secure encryption

- Let F be a keyed function
- $Gen(1^n)$: choose a uniform key $k \in \{0,1\}^n$
- $Enc_k(m)$
  - Choose uniform $r \in \{0,1\}^n$ (IV, Public)
  - Output ciphertext $< r, F_k(r) \oplus m >$
- $Dec_k(c_1, c_2)$: output $c_2 \oplus F_k(c_1)$
- Correctness is immediate


r
key
F
pseudorandom
message
ciphertext

Real-world security?

- What happens if an r is ever reused?
- What is the probability that the r used in some challenge ciphertext is also used for some other ciphertext?
- What happens to the bound if the r is chosen non-uniformly?

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Do Not Do Any Of These Things!

PROS and CONS?

PROS and CONS. Discuss

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PRO If F is a pseudorandom function, then this scheme is CPA-secure

Intuition: If the scheme was not CPA-secure can use to predict F and hence F is not psuedorandom.

PRO Can use same key k for t messages, any t.

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CON Only defined for encryption of n-bit messages

CON $Enc_k(m) = < r, F_k(r) \oplus m >$: n bit message requires $2n$ bits.

CAVEAT Can send long message break up into n-bit chunks.

CON To send t n-bits messages requires $2tn$ bits.

k

$c_1, \ldots, c_t$

k

$m_1, \ldots, m_t$

$c_1 \leftarrow \mathrm{Enc}_k(m_1)$

$\ldots$

$c_t \leftarrow \mathrm{Enc}_k(m_t)$

Sending Many Messages

Goal

The method:

$$Enc_k(m) = < r, F_k(r) \oplus m >$$

is secure but to send ONE n-bit message takes $2n$ bits.

Could send t n-bit messages with $2tn$ bits.

Goal: Send t n-bit message with $< (1+\epsilon)tn$ bits

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securely!

Electronic Code Book (ECB) mode

1. $Enc_k(m_1, \ldots, m_t)$ //note t is arbitrary
   - Send $(F_k(m_1), \ldots, F_k(m_t))$
2. Decryption? Discuss

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4. Drawbacks

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2. Decryption? Discuss
   - Decryption requires F_k to be invertible. Thats fine.
3. To send t n-bit messages, send t n-bit messages. Only tn bits!
4. Drawbacks This is idiotic! Deterministic!

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(I have an iphone)

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If we transmit a picture using ECB here is what Eve sees:

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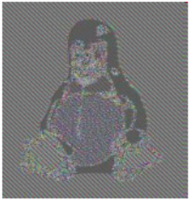
If we transmit a picture using ECB here is what Eve sees:

original



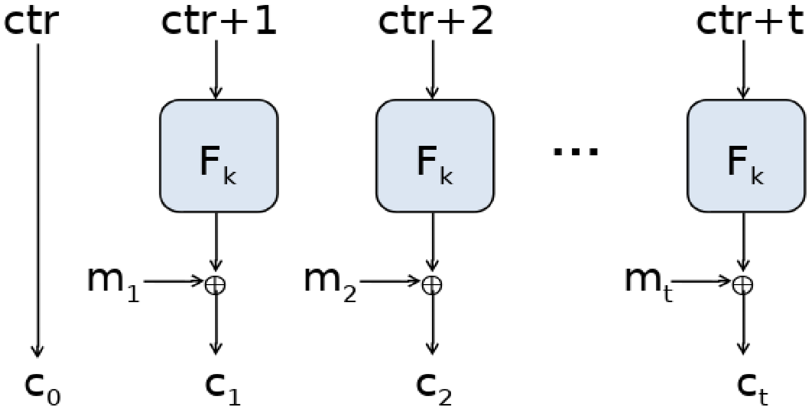
encrypted using ECB mode

(Taken from http://en.wikipedia.org and derived from images created by Larry Ewing (lewing@isc.tamu.edu) using The GIMP.)

Counter (CTR) Mode

- $Enc_k(m_1, \ldots, m_t)$ // note: t is arbitrary
  - Choose $c_0 \leftarrow \{0,1\}^n$
  - For $i = 1$ to t: $c_i = m_i \oplus F_k(c_0 + i \pmod{2^n})$
  - Output $c_0, c_1, \ldots, c_t$
- Decryption? Discuss
- Send t strings by sending one and add to it t times.
- To send t n-bit messages, send $t+1$ n-bit messages.

CTR mode



CTR mode

Theorem: if F is a pseudorandom function, then CTR mode is CPA-secure

Intuition: If CTR is not CPA-secure then can use that to show that to predict F, so F is not pseudorandom.

Cipher Block Chaining (CBC) Mode

- $Enc_k(m_1, \ldots, m_t)$ //note t is arbitrary
  - Choose random $c_0 \leftarrow \{0,1\}^n$ (also called the IV)
  - For $i = 1$ to t: $c_i = F_k(m_i \oplus c_{i-1})$
  - Output $c_0, c_1, \ldots, c_t$
- Decryption? Discuss

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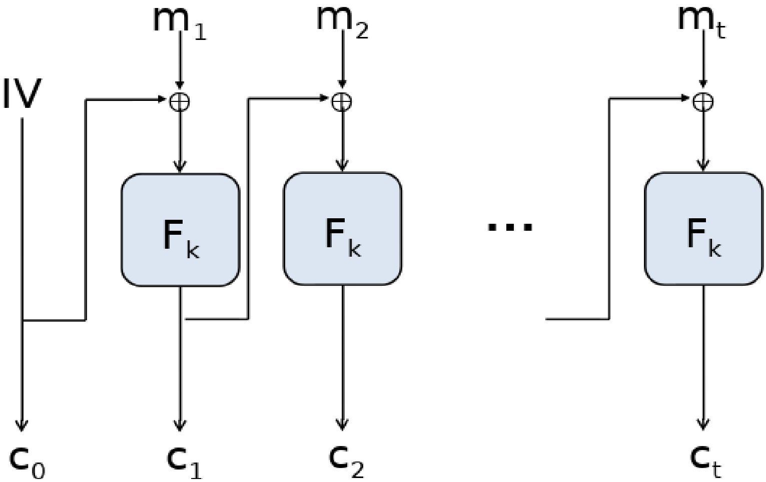
  - Output $c_0, c_1, \ldots, c_t$
- Decryption? Discuss
  - Decryption requires F to be invertible
- Send t strings by sending one and $\oplus$.
- To send t n-bit messages, send $t + 1$ n-bit messages.

CBC mode



CBC mode

Theorem: If F is a pseudorandom permutation, the CBC mode is CPA-secure

Intuition: If CBC is not CPA-secure then can use that to show that to predict F, so F is not pseudorandom.