Public Key Crypto: RSA

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Needed Mathematics- The ϕ Function

Known: If p is prime then $a^{p-1} \equiv 1 \pmod{p}$.
Ramifications: For all m, $a^m \equiv a^{m \pmod{p-1}} \pmod{p}$.
So arithmetic in the exponents is mod $p-1$.

We need to generalize this.

Definition

$\phi(n)$ is the number of numbers in $\{1, \ldots, n-1\}$ that are relatively prime to n.

Note: If p is prime then $\phi(p) = p - 1$.
Known: If n is any number then $a^{\phi(n)} \equiv 1 \pmod{n}$.
Ramifications: For all m, $a^m \equiv a^{m \pmod{\phi(n)}} \pmod{n}$.

Needed Mathematics- Examples

$14^{400} \pmod{1009}$. Repeated squaring takes

$$\lceil \lg(400) \rceil = 9 \text{ steps}$$

$14^{4,000,000,000} \pmod{1009}$. Repeated squaring takes

$$\lceil \lg(4,000,000,000) \rceil = 32 \text{ steps}$$

Can we do better?
$\phi(1009) = 1008$.
$4,000,000,000 \equiv 976 \pmod{1008}$

$$14^{4,000,000,000} \equiv 14^{976} \pmod{1009}$$

Now do repeated squaring which take

$$\lceil \lg(976 \rceil = 10 \text{ steps}$$

More Needed Mathematics

Known: If a, b are relatively prime then $\phi(ab) = \phi(a)\phi(b)$.

Known: Given R, easy to find e rel prime to R and d such that $ed \equiv 1 \pmod{R}$.

Believe: Let $N = pq$, $R = (p-1)(q-1)$ and e rel prime to R. If know N but Not R then hard to find d with $ed \equiv 1 \pmod{R}$.

RSA

Let n be a security parameter

1. Alice picks two primes p, q of length n and computes $N = pq$.
2. Alice computes $\phi(N) = \phi(pq) = (p-1)(q-1)$. Denote by R
3. Alice picks an $e \in \{\frac{R}{3}, \ldots, \frac{2R}{3}\}$ that is relatively prime to R. Alice finds d such that $ed \equiv 1 \pmod{R}$.
4. Alice broadcasts (N, e). (Bob and Eve both see it.)
5. Bob: To send message $m \in \{1, \ldots, N-1\}$, send $m^e \pmod{N}$.
6. If Alice gets $m^e \pmod{N}$ she computes

$$(m^e)^d \equiv m^{ed} \equiv m^{ed \pmod{R}} \equiv m^{1 \pmod{R}} \equiv m$$

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PRO: Alice and Bob can execute the protocol easily.
Biggest PRO: Alice and Bob never had to meet!
Question: Can Eve find out m?

What Do We Really Know about RSA

If Eve can factor then she can crack RSA.

1. Input (N, e) where $N = pq$ and e is rel prime to $R = (p-1)(q-1)$. (p, q, R are NOT part of the input.)
2. Eve factors N to find p, q. Eve computes $R = (p-1)(q-1)$.
3. Eve finds d such that $ed \equiv 1 \pmod{R}$.

Converse is not known to be true!

Definition: Let f be the following function:
Input: N, e where $N = pq$ and e is rel prime to $R = (p-1)(q-1)$ (p, q are NOT in the input).
Output: d such that $ed \equiv 1 \pmod{R}$.

Hardness assumption (HA): f is hard to compute.
VOTE: HA implies RSA secure? YES, NO, UNKNOWN

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The RSA given above is referred to as Plain RSA.
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Eve sees Alice send Bob c_1.

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Scenario:
Eve sees Alice send Bob c_1.
Later Eve sees Alice send Bob c_2.

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That alone makes it insecure.
Plain RSA is never used and should never be used!

PKCS-1.5 RSA

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BAD: To send $m \in \{1, \ldots, N-1\}$, send $m^e \pmod{N}$.

GOOD?: To send $m \in \{1, \ldots, N-1\}$, pick rand r, send $(rm)^e$.
(NOTE- rm means r CONCAT with m here and elsewhere.)

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Still bytes! How to fix? Discuss.

Let $L_1 = \left\lfloor \frac{\lg N}{3} \right\rfloor$, $L_2 = \lfloor \lg N \rfloor - L$.
To send $m \in \{0,1\}^{L_2}$ pick random $r \in \{0,1\}^{L_1}$.
When Alice gets rm she will know that m is the last L_2 bits.

Is PKCS-1.5 RSA Secure?

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- YES (under hardness assumptions and large n)
- NO (there is yet another weird security thing we overlooked)

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(1) will confuse Alice (2) Sealed Bid Scenario.

Final Points About RSA

1. PKCS-2.0-RSA is REALLY used!
2. There are many variants of RSA but all use the ideas above.
3. We may show (much) later show how to prove, assuming the hardness assumption, that RSA is hard to crack.
4. Factoring easy implies RSA crackable. TRUE.
5. RSA crackable implies Factoring easy: UNKNOWN.
6. RSA crackable implies Factoring easy: Often stated in expositions of crypto. They are wrong!
7. Timing attacks on RSA bypass the math.