Quadratic Sieve
Factoring

October 24, 2019
Quick: Factor 8051

Factor 8051. Looks Hard.

$8051 = 90^2 - 7^2 = (90 + 7)(90 - 7) = 97 \times 83$

Key Wrote 8051 as diff of two squares.

General If $N = x^2 - y^2$ then get $N = (x - y)(x + y)$.

But Lucky: we happen to spot two squares that worked.

History Carl Pomerance was on the Math Team in High School and this was a problem he was given. He didn't to solve it in time, but it inspired him to invent the Quadratic Sieve Factoring Algorithm.
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Does this help?
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Does this help? \((81 - 16) \times (81 + 16) = 5 \times 1261\)

\[ 65 \times 97 = 5 \times 1261 \]
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65 divides \(5 \times 1261\), so 65 might share a factor with 1261. Take GCD: \(\text{GCD}(65, 1261) = 13\). So 13 divides 1261.
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**General** If \( (x^2 - y^2) = kN \) then

- \( \text{GCD}(x - y, N) \) might be a nontrivial factor
- \( \text{GCD}(x + y, N) \) might be a nontrivial factor.
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- \( \text{GCD}(x - y, N) \) might be a nontrivial factor
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Want
\[ x^2 - y^2 = kN \]
\[ x^2 - y^2 \equiv 0 \pmod{N} \]
\[ x^2 \equiv y^2 \pmod{N}. \]
Quick: Factor 1649

Want $x^2 \equiv y^2 \pmod{1649}$. Start at $\lceil \sqrt{1649} \rceil = 41$. 

Does any of this help?

$41^2 \equiv 32 = 2^5 \pmod{1649}$

$42^2 \equiv 115 = 5 \times 23 \pmod{1649}$

$43^2 \equiv 200 = 2^3 \times 5^2 \pmod{1649}$

$41^2 \times 43^2 \equiv 2^5 \times 2^3 \times 5^2 = 2^8 \times 5^2 = (2^4 \times 5)^2 = 80^2 \pmod{1649}$

$(41 \times 43)^2 - 80^2 \equiv 0 \pmod{1649}$

$114^2 - 80^2 \equiv 0 \pmod{1649}$

$(114 - 80)(114 + 80) \equiv 34 \times 194 \equiv 0 \pmod{1649}$

$\text{GCD}(34, 1649) = 17$ Found a Factor!
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GCD(34, 1649) = 17 Found a Factor!
Factoring 1649: 97 Also Works?

Recall:

$$(114 - 80)(114 + 80) \equiv 34 \times 194 \equiv 0 \pmod{1649}$$

GCD$(34, 1649) = 17$ Found a Factor!

What if we used 194 instead of 34?

GCD$(194, 1649) = 97$ Found a Factor!

So 194 also works.
Factoring 1649: 97 Also Works?

Recall:

\[(114 - 80)(114 + 80) \equiv 34 \times 194 \equiv 0 \pmod{1649}\]

\[\text{GCD}(34, 1649) = 17 \text{ Found a Factor!}\]
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What is we used 194 instead of 34?

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\text{GCD}(194, 1649) = 97 \quad \text{Found a Factor!}
\]

So 194 also works.
How Can We Make This Happen?

Idea Let $x = \left\lceil \sqrt{N} \right\rceil$.

\[
(x + 0)^2 \equiv y_0 \pmod{N}. \quad \text{Factor } y_0
\]
\[
(x + 1)^2 \equiv y_1 \pmod{N}. \quad \text{Factor } y_1
\]

\[\vdots\]

Look for $I \subseteq \mathbb{N}$ such that:

\[
\prod_{i \in I} y_i = q_1^{2e_1} q_2^{2e_2} \cdots q_k^{2e_k}
\]

and then get

\[
\left( \prod_{i \in I} (x + i) \right)^2 \equiv \left( \prod_{i \in I} q_i^{e_i} \right)^2 \pmod{N}
\]

Let $X = \prod_{i \in I} (x + i)^2$ and $Y = \prod_{i \in I} q_i^{e_i}$.

\[
X^2 - Y^2 \equiv 0 \pmod{N}.
\]

Is this a good idea? Discuss.
Look at the First Step

\[(x + 0)^2 \equiv y_0 \pmod{N}. \quad \text{Factor } y_0\]
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In order to factor \( N \) we needed to factor the \( y_i \)'s.
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In order to factor \( N \) we needed to factor the \( y_i \)'s. Really? Darn! Ideas?
**B-Factoring**

Idea $B$ be a parameter. $p_1 < p_2 < \cdots < p_B$ are the first $B$ primes.

Def A number is $B$-factored if its largest prime factor is $\leq p_B$. 

Example $B = 5$. Primes $2, 3, 5, 7, 11$. $1000 = 2^3 \times 5^3$. So $B$-factored. 

$27378897 = 11 \times 35557$. $35557$ is composite. NOT $B$-factored.

Is $B$-factoring faster than factoring? Let's try to $B$-factor $82203$.

1. Divide 2 into it. 2 does not divide 82203.
2. Divide 3 into what's left. $82203 = 3 \times 27401$.
3. Divide 5 into what's left. 5 does not divide 27401.
4. Divide 7 into what's left. 7 does not divide 27401.
5. Divide 11 into what's left. $82203 = 3 \times 11 \times 7473$.
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6. DONE.  NOT $B$-factorable.  Only did $B$ divisions.
Abbreviation

We use $B$-fact for $B$-factorable.

Why?
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We use $B$-fact for $B$-factorable.

Why?

Space on slides!
Example of Algorithm that Uses $B$-Factoring

Want to factor 539873. $B = 7$ so use 2, 3, 5, 7, 11, 13, 17

$\left\lfloor \sqrt{539873} \right\rfloor = 735$
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Want to factor 539873. $B = 7$ so use 2, 3, 5, 7, 11, 13, 17

$\lceil \sqrt{539873} \rceil = 735$

$735^2 \equiv 352 = 2^5 \times 11 \pmod{539873}$.

$736^2, \ldots, 749^2$ did not 7-factor.
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$750^2 \equiv 22627 \equiv 11^3 \times 17 \pmod{539873}$. 
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751$^2$,\ldots, 782$^2$ did not 7-factor.
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Example of Algorithm that Uses $B$-Factoring

Want to factor 539873. $B = 7$ so use 2, 3, 5, 7, 11, 13, 17

\[ \sqrt[5]{539873} \approx 735 \]

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751$^2, \ldots, 782^2$ did not 7-factor.

783$^2 \equiv 22627 \equiv 11^3 \times 17 \pmod{539873}$.

784$^2, \ldots, 800^2$ did not 7-factor.

801$^2 \equiv 101728 \equiv 2^5 \times 11 \times 17^2 \pmod{539873}$.

Can we use this? Next Slide I write it nicer.
Example Continued: Trying to factor 539873

\[ 735^2 \equiv 352 = 2^5 \times 11 \pmod{539873}. \]
\[ 750^2 \equiv 22627 \equiv 11^3 \times 17 \pmod{539873}. \]
\[ 783^2 \equiv 22627 \equiv 11^3 \times 17 \pmod{539873}. \]
\[ 801^2 \equiv 101728 \equiv 2^5 \times 11 \times 17^2 \pmod{539873}. \]

Can you find a way to multiple some of these to get \( X^2 \equiv Y^2? \)
Example Continued: Trying to factor 539873

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\[783^2 \equiv 22627 \equiv 11^3 \times 17 \pmod{539873}.
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\[801^2 \equiv 101728 \equiv 2^5 \times 11 \times 17^2 \pmod{539873}.
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Can you find a way to multiple some of these to get \(X^2 \equiv Y^2\)?

\[(735 \times 801)^2 \equiv 2^{10} \times 11^2 \times 17^2
\]
\[(735 \times 801)^2 \equiv (2^5 \times 11 \times 17)^2
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Example Finished: Trying to factor 539873

We have found:

\[ 48862^2 - 5984^2 \equiv 0 \pmod{539873} \]

Now we use it to find a factor:
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Now we use it to find a factor:

\[ (48862 - 5984) \times (48862 + 5984) \equiv 0 \pmod{539873} \]

GCD(42878, 539873) = 1949

1949 divides 539873. Found a Factor!
Example Finished: Trying to factor 539873

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\[42878 \times 54846 \equiv 0 \pmod{539873}\]

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\[ 42878 \times 54846 \equiv 0 \pmod{539873} \]

\[ \text{GCD}(42878, 539873) = 1949 \]

1949 divides 539873. **Found a Factor!**
We Noticed That... Can a Program?

\[
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Notice that
\[
(735 \times 801)^2 \equiv 2^{10} \times 11^2 \times 17^2
\]

How can a program Notice That?
What is a program supposed to notice? Discuss.
We Noticed That... Can a Program? Cont

\[
\sqrt{539873} = 735
\]

\[735^2 \equiv 352 = 2^5 \times 11 \pmod{539873}.\]

\[750^2 \equiv 22627 \equiv 11^3 \times 17 \pmod{539873}.\]

\[783^2 \equiv 22627 \equiv 11^3 \times 17 \pmod{539873}.\]

\[801^2 \equiv 101728 \equiv 2^5 \times 11 \times 117^2 \pmod{539873}.\]

\[(735 \times 801)^2 \equiv 2^{10} \times 11^2 \times 17^2\]

All of the exponents on the right-hand-side are even.
We Noticed That... Can a Program? Cont

\[ \left\lfloor \sqrt{539873} \right\rfloor = 735 \]

\[ 735^2 \equiv 352 \equiv 2^5 \times 11 \pmod{539873}. \]

\[ 750^2 \equiv 22627 \equiv 11^3 \times 17 \pmod{539873}. \]

\[ 783^2 \equiv 22627 \equiv 11^3 \times 17 \pmod{539873}. \]

\[ 801^2 \equiv 101728 \equiv 2^5 \times 11 \times 17^2 \pmod{539873}. \]

\[ (735 \times 801)^2 \equiv 2^{10} \times 11^2 \times 17^2 \]

All of the exponents on the right-hand-side are even.

We want to find a set of right-hand-sides so that when multiplied together all of the exponents are even.
Idea One

\[ \sqrt[539873]{539873} \equiv 735 \]

\[
\begin{align*}
735^2 & \equiv 352 & \equiv & 2^5 \times 11^1 & (5,0,0,0,11,0,0) \\
750^2 & \equiv 22627 & \equiv & 11^3 \times 17^1 & (0,0,0,0,3,0,1) \\
783^2 & \equiv 73216 & \equiv & 2^9 \times 11^1 \times 13^1 & (9,0,0,0,1,1,0) \\
801^2 & \equiv 101728 & \equiv & 2^5 \times 11^1 \times 17^2 & (5,0,0,0,11,0,2)
\end{align*}
\]

Want some combination of the vectors to have all even numbers. Can we use Linear Algebra? Discuss
Idea One


\[ \left\lceil \sqrt{539873} \right\rceil = 735 \]

\[
\begin{align*}
735^2 & \equiv 352 \equiv 2^5 \times 11^1 & (5, 0, 0, 0, 11, 0, 0) \\
750^2 & \equiv 22627 \equiv 11^3 \times 17^1 & (0, 0, 0, 0, 3, 0, 1) \\
783^2 & \equiv 73216 \equiv 2^9 \times 11^1 \times 13^1 & (9, 0, 0, 0, 1, 1, 0) \\
801^2 & \equiv 101728 \equiv 2^5 \times 11^1 \times 17^2 & (5, 0, 0, 0, 11, 0, 2)
\end{align*}
\]

Want some combination of the vectors to have all even numbers.
Can we use Linear Algebra? Discuss

We do not need the numbers. All we need are the parities!
Idea Two

Store parities of exponents in vector.

$\left\lceil \sqrt{539873} \right\rceil = 735$

$735^2 \equiv 352 \equiv 2^5 \times 11^1 (1, 0, 0, 0, 1, 0, 0)$

$750^2 \equiv 22627 \equiv 11^3 \times 17^1 (0, 0, 0, 0, 1, 0, 1)$

$783^2 \equiv 73216 \equiv 2^9 \times 11^1 \times 13^1 (1, 0, 0, 0, 1, 1, 0)$

$801^2 \equiv 101728 \equiv 2^5 \times 11^1 \times 17^2 (1, 0, 0, 0, 1, 0, 0)$
Idea Two

Store parities of exponents in vector.
\[\lceil \sqrt{539873} \rceil = 735\]

\[
\begin{align*}
735^2 & \equiv 352 \equiv 2^5 \times 11^1 \quad (1,0,0,0,1,0,0) \\
750^2 & \equiv 22627 \equiv 11^3 \times 17^1 \quad (0,0,0,0,1,0,1) \\
783^2 & \equiv 73216 \equiv 2^9 \times 11^1 \times 13^1 \quad (1,0,0,0,1,1,0) \\
801^2 & \equiv 101728 \equiv 2^5 \times 11^1 \times 17^2 \quad (1,0,0,0,1,0,0)
\end{align*}
\]

Well Defined Math Problem Given a set of 0-1 $B$-vectors over $\mathbb{Z}_2$, does some subset of them sum to $\vec{0}$? Equivalent to asking if some subset is linearly dependent.

- Can solve using Gaussian Elimination.
- If there are $B + 1$ vectors then there will be such a set.
Quad Sieve Alg: First Attempt

Given $N$ let $x = \lceil \sqrt{N} \rceil$. All $\equiv$ are mod $N$. $B, M$ are params.
Quad Sieve Alg: First Attempt

Given $N$ let $x = \left\lceil \sqrt{N} \right\rceil$. All $\equiv$ are mod $N$. $B, M$ are params.

$(x + 0)^2 \equiv y_0$ Try to $B$-Factor $y_0$ to get parity $\vec{v}_0$

$\vdots$

$(x + M)^2 \equiv y_M$ Try to $B$-Factor $y_M$ to get parity $\vec{v}_M$
Quad Sieve Alg: First Attempt

Given $N$ let $x = \lceil \sqrt{N} \rceil$. All $\equiv$ are mod $N$. $B, M$ are params.

$(x + 0)^2 \equiv y_0$ Try to $B$-Factor $y_0$ to get parity $\vec{v}_0$

\[ \vdots \]

$(x + M)^2 \equiv y_M$ Try to $B$-Factor $y_M$ to get parity $\vec{v}_M$

Some of the $y_i$ were $B$-factored, but some were not.
Let $I$ be the set of all $i$ such that $y_i$ was $B$-factored.
Quad Sieve Alg: First Attempt

Given $N$ let $x = \left\lceil \sqrt{N} \right\rceil$. All $\equiv$ are mod $N$. $B, M$ are params.

\[(x + 0)^2 \equiv y_0 \quad \text{Try to $B$-Factor } y_0 \text{ to get parity } \vec{v}_0 \]
\[\vdots \]
\[(x + M)^2 \equiv y_M \quad \text{Try to $B$-Factor } y_M \text{ to get parity } \vec{v}_M \]

Some of the $y_i$ were $B$-factored, but some were not.
Let $I$ be the set of all $i$ such that $y_i$ was $B$-factored.

Find $J \subseteq I$ such that $\sum_{i \in J} \vec{v}_i = \vec{0}$.
Quad Sieve Alg: First Attempt

Given $N$ let $x = \lceil \sqrt{N} \rceil$. All $\equiv$ are mod $N$. $B, M$ are params.

$$(x + 0)^2 \equiv y_0 \quad \text{Try to } B\text{-Factor } y_0 \text{ to get parity } \vec{v}_0$$

\[ \vdots \]

$$(x + M)^2 \equiv y_M \quad \text{Try to } B\text{-Factor } y_M \text{ to get parity } \vec{v}_M$$

Some of the $y_i$ were $B$-factored, but some were not. Let $I$ be the set of all $i$ such that $y_i$ was $B$-factored.

Find $J \subseteq I$ such that $\sum_{i \in J} \vec{v}_i = \vec{0}$.

Hence $\prod_{i \in J} y_i$ has all even exponents. Hence:

$$\prod_{i \in J} y_i = q_1^{2e_1} q_2^{2e_2} \cdots q_k^{2e_k}$$
Quad Sieve Alg: First Attempt, Cont

\[ \prod_{i \in J} y_i = q_1^{2e_1} q_2^{2e_2} \cdots q_k^{2e_k} \]

\[ \left( \prod_{i \in J} (x + i) \right)^2 \equiv \left( \prod_{i \in J} q_i^{e_i} \right)^2 \pmod{N} \]

Let \( X = \prod_{i \in J} (x + i) \) and \( Y = \prod_{i \in J} q_i^{e_i} \).

\[ X^2 - Y^2 \equiv 0 \pmod{N}. \]

\[ (X - Y)(X + Y) = kN \text{ for some } k \]

\( \gcd(X - Y, N) \), \( \gcd(X + Y, N) \) should yield factors.
What Could go Wrong

1. There is no set of rows that is linearly dependent.
2. You find \( X, Y \) such that \( X^2 \equiv Y^2 \mod N \) but then \( \text{GCD}(X - Y, N) = 1 \) and \( \text{GCD}(X + Y, N) = N \). This is very rare so we will not worry about it.
What Could go Wrong

1. There is no set of rows that is linearly dependent.
What Could go Wrong

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2. You find $X, Y$ such that $X^2 \equiv Y^2 \mod N$ but then $\gcd(X - Y, N) = 1$ and $\gcd(X + Y, N) = N$. This is very rare so we will not worry about it.
Balancing Act

1. Run time will depend on $B$ and $M$. Gaussian Elimination is $O(B^3)$, which will be the main time sink. So want $B$ small.

2. If $B$ is large, then more numbers are $B$-fact, so have to go through less numbers to get $B+1$ $B$-fact numbers (hence $B+1$ vectors of dim $B$), so guaranteed to have a linear dependency. Hence want $B$ large.

3. In practice $B$ is chosen carefully based on computation and conjectures in Number Theory.
Balancing Act

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3. In practice $B$ is chosen carefully based on computation and conjectures in Number Theory.
Most Important Step to Speed Up

An earlier slide said

Gaussian Elimination is $O(B^3)$ which will be the main time sink.
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What about $B$ factoring $M$ numbers. That would seem to also be a time sink.
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Gaussian Elimination is $O(B^3)$ which will be the main time sink.

What about $B$ factoring $M$ numbers. That would seem to also be a time sink.

The key to making the algorithm practical is Carl Pomerance’s insight which is the how to do all that $B$-factoring fast. Do do this we need a LOOOOOONG aside on Sieving.
Finding all Primes \( \leq 48 \), the Stupid Way

To find all primes \( \leq 48 \) we could do the following:

\[
\text{for } i = 2 \text{ to } 48 \text{ if } \text{isprime}(i) = \text{YES} \text{ then output } i.
\]

Is this a good idea? Discuss.
Finding all Primes $\leq 48$, the Stupid Way

To find all primes $\leq 48$ we could do the following:

$$\text{for } i = 2 \text{ to } 48 \text{ if } \text{isprime}(i) = \text{YES} \text{ then output } i.$$ 

Is this a good idea? Discuss.

No You are testing many numbers that you could have, ahead of time, ruled out.
Finding all primes \( \leq 48 \) the Smart Way

Write down the numbers \( \leq 48 \).

\[
\begin{array}{cccccccccccc}
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\
28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 \\
40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\
\end{array}
\]

Now output first unmarked—2—and MARK all multiples of 2.
### Finding all primes $\leq 48$ the Smart Way

Write down the numbers $\leq 48$.

```
2  3  4  5  6  7  8  9  10  11  12  13  14  15

16 17 18 19 20 21 22 23 24 25 26 27

28 29 30 31 32 33 34 35 36 37 38 39

40 41 42 43 44 45 46 47 48
```

Now output first unmarked—2—and MARK all multiples of 2.
We Have Marked Multiples of 2

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We Have Marked Multiples of 2

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Now output first unmarked—3—and MARK all multiples of 3.
We Have Marked Multiples of 2 and 3

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We Have Marked Multiples of 2 and 3

Now Have:

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| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10| 11| 12| 13| 14| 15|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
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| 16| 17| 18| 19| 20| 21| 22| 23| 24| 25| 26| 27|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
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| 28| 29| 30| 31| 32| 33| 34| 35| 36| 37| 38| 39|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
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| 40| 41| 42| 43| 44| 45| 46| 47| 48|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| X | X | X | X | X | X | X | X | X |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

Now output first unmarked—5—and MARK all multiples of 5.
We Have Marked Multiples of 2, 3 and 5

Now Have:

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Now output first unmarked—7—and MARK all multiples of 7. You get the idea so we stop here.
We Have Marked Multiples of 2, 3 and 5

Now Have:

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Now output first unmarked—7—and MARK all multiples of 7. You get the idea so we stop here.
A Few Points About this Process

Speed

1. This process is really fast since when (say) MARKING mults of 3: We DO NOT look at (say) 23 and say no. WE DO NOT look at (say) 23 at all.
2. The KEY to many Number Theory Algorithms is not looking
3. Good number theory algs act on a need-to-know basis.
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3. Good number theory algs act on a need-to-know basis.

Could we make it faster?
1. When MARKING mults of 3 we could mark 3, 3+6, 3 + 2 × 6 since mults of 2 are already MARKED.
2. When MARKING mults of 5 we could mark 5, 5+10, 5 + 2 × 10 since mults of 2 are already MARKED. Hard to also avoid mults of 3: 5, 25, 35 not equally spaced.
3. When MARKING mults of BLAH we could BLAHBLAH.
4. If our goal was to JUST get a list of primes, we might do this.
5. Our goal will be to FACTOR these numbers. As such we cannot use this shortcut. (Clear later.)
The Sieve of Eratosthenes

1. Input($N$)
2. Write down 2, 3, . . ., $N$. All are unmarked.
3. (MARK STEP) Goto the first unmarked element of the list $p$. Output($p$). Keep pointer there. (When pointer is at $N$ or beyond then stop.)
4. Mark all multiples of $p$ up to $\left\lfloor \frac{N}{p} \right\rfloor p$. (This takes $\frac{N}{p}$ steps.)
5. GOTO MARK STEP.

Time:

$$\sum_{p \leq N} \frac{N}{p} = N \sum_{p \leq N} \frac{1}{p}$$

New Question: What is $\sum_{p \leq N} \frac{1}{p}$?
As Aside on $\sum_{p \leq N} \frac{1}{p}$

October 24, 2019
Notation

\[
\sum_{n \leq N} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N}
\]

\[
\sum_{n < \infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots
\]

\[
\sum_{p \leq N} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{q}
\]

where \( q \) is the largest prime \( \leq N \).

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\[ \sum_{p < \infty} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \]

Example

\[ \sum_{p \leq 14} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} \]
What is $\sum_{p \leq N} \frac{1}{p}$ Asymptotically? History

When I looked up $\sum_{p \leq N} \frac{1}{p}$ on the web I found:

1. Proofs that $\sum_{p \leq N} \frac{1}{p}$ diverges.
2. Some of those proofs show that $\sum_{p \leq N} \frac{1}{p} \geq \ln(\ln(N)) + O(1)$.
3. Nothing on upper bounds on the sum.
4. TA Erik, when proofreading these slides, was able to find the theorem, though it was difficult. It's Merten's Second Thm.

A sequence of events:
1. In 2010 Larry W showed Bill G a proof that $\sum_{p \leq N} \frac{1}{p} \leq \ln(\ln(N)) + O(1)$.
2. Larry says its a well known theorem but never written down.
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Moral of the Story Google is not always enough.
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Moral of the Story Google is not always enough.
More on \( \sum_{p \leq N} \frac{1}{p} \)

1. \( \sum_{n \leq N} \frac{1}{n} \sim \ln(n) \).
2. \( \sum_{p \leq N} \frac{1}{p} \sim \ln(\ln(N)) \)

How good is this approximation?

1) When \( N \geq 286 \),

\[
\ln(\ln(N)) - \frac{1}{2(\ln(N))^2} + C \leq \sum_{p \leq N} \frac{1}{p} \leq \ln(\ln(N)) + \frac{1}{(2\ln(N))^2} + C,
\]

where \( C \sim 0.261497212847643 \).

2)

\[
\begin{align*}
\sum_{p \leq 10} \frac{1}{p} &= 1.176 \\
\sum_{p \leq 10^9} \frac{1}{p} &= 3.293 \\
\sum_{p \leq 10^{100}} \frac{1}{p} &\sim 5.7 \\
\sum_{p \leq 10^{1000}} \frac{1}{p} &\sim 7.8
\end{align*}
\]
Take Away

\[
\sum_{p \leq N} \frac{1}{p} \sim \ln(\ln N)
\]

▶ This is a very good approximation.
▶ This is very small
▶ (Cheating to make math easier) The largest \(pq\) factored is around 170-digits. We assume a limit of 1000 digits. Hence we treat \(\ln(\ln(N))\) as if it was

\[
\ln(\ln(N)) \leq \ln(\ln(1000)) \sim 8.
\]

(Nobody else does this.)
Back to our Aside on Sieves

October 24, 2019
Time Analysis of Sieve of E

The Sieve of E can find all primes \( \leq N \) in time

\[
\leq N \sum_{p \leq N} \frac{1}{p} \leq N \ln(\ln(N))
\]
Time Analysis of Sieve of E

The Sieve of E can find all primes \( \leq N \) in time

\[
\leq N \sum_{p \leq N} \frac{1}{p} \leq N \ln(\ln(N))
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How long would finding all primes \( \leq N \) be the stupid way?

\[
\sum_{n \leq N} \ln(n) \sim N \ln(N)
\]
Time Analysis of Sieve of E

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\sum_{n \leq N} \ln(n) \sim N \ln(N)
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▶ The time difference here is not that impressive. When we modify the Sieve to actually factor, it will be much more impressive.
Time Analysis of Sieve of E

The Sieve of E can find all primes $\leq N$ in time

$$\leq N \sum_{p \leq N} \frac{1}{p} \leq N \ln(\ln(N))$$

How long would finding all primes $\leq N$ be the stupid way?

$$\sum_{n \leq N} \ln(n) \sim N \ln(N)$$

- The time difference here is not that impressive. When we modify the Sieve to actually factor, it will be much more impressive.
- The key to the speed of The Sieve of E is that when it marks it DOES NOT look at (say) 3 and say Oh, thats not even. It literally does not look at all!
B-Factoring-Variant on Sieve of E: Example

The Sieve of E marked all evens.  
**Better** Divide by 2 knowing it will work. Then divide by 2 again (it might not work) until factor out all powers of 2.

The Sieve of E marked all numbers ≡ 0 (mod 3)  
**Better** Divide by 3 knowing it will work. Then divide by 3 again (it might not work) until factor out all powers of 3.

Do this for the first $B$ primes and you will have $B$-factored many numbers.
B-factoring all $N \leq 48$, the Smart Way

Write down numbers $\leq 48$. We 2-factor them, so divide by 2, 3.

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First unmarked is 2. DIVIDE mults of 2 by 2.
## Divide by 2

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First unmarked is 2. DIVIDE mults of 3 by 3.
We only show the last row (for reasons of space).

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<td>$2^3 \times 5$</td>
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- 48 was 2-factored
- Nothing else was.
Variant of The Sieve of Eratosthenes: Algorithm

1. Input\((N, B)\)
2. Write down 2, 3, \ldots, N. All are have blank in box.
3. (BOX STEP) Goto the first blank box, \(p\). (When have visited this step \(B\) times then stop).
4. Divide what the elements \(p, 2p, \ldots, \left\lfloor \frac{N}{p} \right\rfloor p\) by \(p\) then \(p\) again and again until can’t. (This takes \(\sim \frac{N}{p}\) steps.)
5. GOTO BOX STEP.

Time:

\[
\sum_{p \leq B} \frac{N}{p} + \sum_{p \leq B} \frac{N}{p^2} + \sum_{p \leq B} \frac{N}{p^3} + \sum_{p \leq B} \frac{N}{p^4} \cdots
\]

\[
= N \left( \sum_{p \leq B} \frac{1}{p} + \sum_{p \leq B} \frac{1}{p^2} + \sum_{p \leq B} \frac{1}{p^3} + \sum_{p \leq B} \frac{1}{p^4} + \cdots \right)
\]
Variant of The Sieve of Eratosthenes: Analysis

\[
N \left( \sum_{p \leq B} \frac{1}{p} + \sum_{p \leq B} \frac{1}{p^2} + \sum_{p \leq B} \frac{1}{p^3} + \sum_{p \leq B} \frac{1}{p^4} + \cdots \right)
\]

\[
N \sum_{p \leq B} \frac{1}{p} + N \sum_{p \leq B} \frac{1}{p^2} + N \sum_{p \leq B} \frac{1}{p^3} + N \sum_{p \leq B} \frac{1}{p^4} + \cdots
\]

\[
= N \ln(\ln(B)) + N \sum_{a=2}^{\infty} \sum_{p \leq B} \frac{1}{p^a}
\]

Next slide shows that \( N \sum_{a=2}^{\infty} \sum_{p \leq B} \frac{1}{p^a} \leq (0.5)N \), so time is
Variant of The Sieve of Eratosthenes: Analysis

\[= N \left( \sum_{p \leq B} \frac{1}{p} + \sum_{p \leq B} \frac{1}{p^2} + \sum_{p \leq B} \frac{1}{p^3} + \sum_{p \leq B} \frac{1}{p^4} + \cdots \right)\]

\[= N \sum_{p \leq B} \frac{1}{p} + N \sum_{p \leq B} \frac{1}{p^2} + N \sum_{p \leq B} \frac{1}{p^3} + N \sum_{p \leq B} \frac{1}{p^4} + \cdots\]

\[= N \ln(\ln(B)) + N \sum_{a=2}^{\infty} \sum_{p \leq B} \frac{1}{p^a}\]

Next slide shows that \(N \sum_{a=2}^{\infty} \sum_{p \leq B} \frac{1}{p^a} \leq (0.5)N\), so time is

\[\leq N \ln(\ln(B)) + (0.5)N.\]

Note: The mult constants really are \(\leq 1\) and it does matter for real world performance.
Variant of The Sieve of E: That last term is $\leq N$

$$= N \sum_{a=2}^{\infty} \sum_{p \leq B} \frac{1}{p^a} = N \sum_{p \leq B} \sum_{a=2}^{\infty} \frac{1}{p^a}$$
Variant of The Sieve of E: That last term is $\leq N$

$$
\begin{align*}
= N \sum_{a=2}^{\infty} \sum_{p \leq B} \frac{1}{p^a} &= N \sum_{p \leq B} \sum_{a=2}^{\infty} \frac{1}{p^a} \\
&= N \sum_{p \leq B} \frac{1/p^2}{1 - (1/p)} \\
&= N \sum_{p \leq B} \frac{1}{p^2 - p} \sim N \sum_{p \leq B} \frac{1}{p^2}
\end{align*}
$$

How big is $\sum_{p \leq B} \frac{1}{p^2}$?
Variant of The Sieve of E: That last term is $\leq N$

$$= N \sum_{a=2}^{\infty} \sum_{p \leq B} \frac{1}{p^a} = N \sum_{p \leq B} \sum_{a=2}^{\infty} \frac{1}{p^a}$$

$$= N \sum_{p \leq B} \frac{1/p^2}{1 - (1/p)}$$

$$= N \sum_{p \leq B} \frac{1}{p^2 - p} \sim N \sum_{p \leq B} \frac{1}{p^2}$$

How big is $\sum_{p \leq B} \frac{1}{p^2}$?

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ cvg. Do you know to what?
Variant of The Sieve of E: That last term is $\leq N$

$$
\sum_{a=2}^{\infty} \sum_{p \leq B} \frac{1}{p^a} = \sum_{p \leq B} \sum_{a=2}^{\infty} \frac{1}{p^a} = \sum_{p \leq B} \frac{1}{p^2} \frac{1}{1 - (1/p)} = \sum_{p \leq B} \frac{1}{p^2 - p} \sim \sum_{p \leq B} \frac{1}{p^2}
$$

How big is $\sum_{p \leq B} \frac{1}{p^2}$?

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ cvg. Do you know to what? $\frac{\pi^2}{6} \sim 1.644$
Variant of The Sieve of E: That last term is \( \leq N \)

\[
= N \sum_{a=2}^{\infty} \sum_{p \leq B} \frac{1}{p^a} = N \sum_{p \leq B} \sum_{a=2}^{\infty} \frac{1}{p^a}
\]

\[
= N \sum_{p \leq B} \frac{1}{p^2 - (1/p)}
\]

\[
= N \sum_{p \leq B} \frac{1}{p^2} - p \sim N \sum_{p \leq B} \frac{1}{p^2}
\]

How big is \( \sum_{p \leq B} \frac{1}{p^2} \)?

1. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) cvg. Do you know to what? \( \frac{\pi^2}{6} \sim 1.644 \)
2. \( \sum_{p=1}^{\infty} \frac{1}{p^2} \) cvg. Do you know to what?
Variant of The Sieve of E: That last term is $\leq N$

$$\begin{align*}
= N \sum_{a=2}^{\infty} \sum_{p \leq B} \frac{1}{p^a} &= N \sum_{p \leq B} \sum_{a=2}^{\infty} \frac{1}{p^a} \\
= N \sum_{p \leq B} \frac{1}{p^2} \frac{1}{1 - (1/p)} &= N \sum_{p \leq B} \frac{1}{p^2 - p} \sim N \sum_{p \leq B} \frac{1}{p^2}
\end{align*}$$

How big is $\sum_{p \leq B} \frac{1}{p^2}$?

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ cvg. Do you know to what? $\frac{\pi^2}{6} \sim 1.644$

2. $\sum_{p=1}^{\infty} \frac{1}{p^2}$ cvg. Do you know to what? $\sim 0.45$. 
Recap Variant of The Sieve of Eratosthenes

Given $N, B$ can $B$-factor $\{2, \ldots, N\}$ in time

\[ N \leq \ln(\ln(B)) + 0.5N \]

Can easily modify to get a fast algorithm for $B$-factoring $N_1, \ldots, N_1 + N$.

This is not the problem we originally needed to solve, though its close. We now go back to our original problem.
Back to Quadratic Sieve
Factoring Algorithm

October 24, 2019
Recall Quad Sieve Alg: First Attempt

Given $N$ let $x = \lceil \sqrt{N} \rceil$. All $\equiv$ are mod $N$. $B, M$ are params.

$(x + 0)^2 \equiv y_0 \quad$ Try to $B$-Factor $y_0$ to get parity $\vec{v}_0$

$\vdots$

$(x + M)^2 \equiv y_M \quad$ Try to $B$-Factor $y_M$ to get parity $\vec{v}_M$

Let $I \subseteq \{0, \ldots, M\}$ so that $(\forall i \in I)\), $y_i$ is $B$-factored. Find $J \subseteq I$ such that $\sum_{i \in J} \vec{v}_i = 0$. Hence $\prod_{i \in J} y_i$ has all even exponents.

$$\prod_{i \in J} y_i = q_1^{2e_1} q_2^{2e_2} \cdots q_k^{2e_k}$$

$$(\prod_{i \in J} (x + i))^2 \equiv (\prod_{i \in J} q_i^{e_i})^2 \pmod{N}$$

Let $X = \prod_{i \in J} (x + i)$ and $Y = \prod_{i \in J} q_i^{e_i}$.

$$X^2 - Y^2 \equiv 0 \pmod{N}.$$ 

$\text{GCD}(X - Y, N), \text{GCD}(X + Y, N)$ should yield factors.
Given $N$ let $x = \left\lceil \sqrt{N} \right\rceil$. All $\equiv$ are mod $N$. $B, M$ are params.

$(x + 0)^2 \equiv y_0$ Try to $B$-Factor $y_0$ to get parity $\vec{v}_0$

\vdots

$(x + M)^2 \equiv y_M$ Try to $B$-Factor $y_M$ to get parity $\vec{v}_M$

How do we $B$-factor all of those numbers?
Recall Quad Sieve Alg: First Attempt, First Step

Given \( N \) let \( x = \left\lceil \sqrt{N} \right\rceil \). All \( \equiv \) are mod \( N \). \( B, M \) are params.

\[(x + 0)^2 \equiv y_0 \quad \text{Try to } B\text{-Factor } y_0 \text{ to get parity } \vec{v}_0\]

\[\vdots \]

\[(x + M)^2 \equiv y_M \quad \text{Try to } B\text{-Factor } y_M \text{ to get parity } \vec{v}_M\]

How do we \( B \)-factor all of those numbers?
Modified Sieve of E \( B \)-factored \( N_1 + 1, \ldots, N_1 + N \).
Recall Quad Sieve Alg: First Attempt, First Step

Given $N$ let $x = \left\lceil \sqrt{N} \right\rceil$. All $\equiv$ are mod $N$. $B, M$ are params.

$(x + 0)^2 \equiv y_0$  Try to $B$-Factor $y_0$ to get parity $\vec{v}_0$

$\vdots$

$(x + M)^2 \equiv y_M$  Try to $B$-Factor $y_M$ to get parity $\vec{v}_M$

How do we $B$-factor all of those numbers?
Modified Sieve of E $B$-factored $N_1 + 1, \ldots, N_1 + N$.
We need to $B$-factor $y_0, y_1, \ldots, y_M$. 
Recall Quad Sieve Alg: First Attempt, First Step

Given $N$ let $x = \lceil \sqrt{N} \rceil$. All $\equiv$ are mod $N$. $B, M$ are params.

$$(x + 0)^2 \equiv y_0 \quad \text{Try to } B\text{-Factor } y_0 \text{ to get parity } \vec{v}_0$$

$$\vdots$$

$$(x + M)^2 \equiv y_M \quad \text{Try to } B\text{-Factor } y_M \text{ to get parity } \vec{v}_M$$

How do we $B$-factor all of those numbers?
Modified Sieve of E $B$-factored $N_1 + 1, \ldots, N_1 + N$.
We need to $B$-factor $y_0, y_1, \ldots, y_M$.

Plan It was more efficient to $B$-factor $2, \ldots, N$ all at once then one at at time. Same will be true for $y_0, \ldots, y_M$. 
New Problem Given $N, B, M, x$, want to $B$-factor
$(x + 0)^2 \pmod{N}$
$(x + 1)^2 \pmod{N}$
$\vdots \vdots$
$(x + M)^2 \pmod{N}$
We do an example on the next slide.
The Quadratic Sieve: Example

\[ N = 1147, \ B = 2, \ M = 10, \ x = 34. \]

Want to 2-factor (so all powers of 2 and 3)
\((34 + 0)^2 \pmod{1147}\)
\[
\vdots \quad \vdots \quad \vdots
\]
\((34 + 10)^2 \pmod{1147}\)
The Quadratic Sieve: Example

\[ N = 1147, \ B = 2, \ M = 10, \ x = 34. \]
Want to 2-factor (so all powers of 2 and 3)
\[(34 + 0)^2 \pmod{1147}\]
\[\vdots \vdots \vdots \]
\[(34 + 10)^2 \pmod{1147}\]
For the Sieve of E when we wanted to divide by \( p \) we looked at every \( p \)th element. Is there an analog here?
The Quadratic Sieve: Example

\[ N = 1147, \ B = 2, \ M = 10, \ x = 34. \]
Want to 2-factor (so all powers of 2 and 3)
\[(34 + 0)^2 \pmod{1147}\]
\[
\vdots \quad \vdots \quad \vdots
\]
\[(34 + 10)^2 \pmod{1147}\]
For the Sieve of E when we wanted to divide by \( p \) we looked at every \( p \)th element. Is there an analog here?
For which \( 0 \leq i \leq 10 \) does 2 divide \((34 + i)^2 \pmod{1147}\)?
The Quadratic Sieve: Example

\[ N = 1147, \ B = 2, \ M = 10, \ x = 34. \]
Want to 2-factor (so all powers of 2 and 3)
\[ (34 + 0)^2 \pmod{1147} \]
\[ \vdots \]
\[ (34 + 10)^2 \pmod{1147} \]
For the Sieve of E when we wanted to divide by \( p \) we looked at every \( p \)th element. Is there an analog here?

For which \( 0 \leq i \leq 10 \) does 2 divide \( (34 + i)^2 \pmod{1147} \)?
Next Slide
The Quadratic Sieve: Example of dividing by 2

Need to know the set of $0 \leq i \leq 10$ such that 2 divides

$((34 + i)^2 \pmod{1147})$

What is $(34 + i)^2 \pmod{1147}$?

Since $0 \leq i \leq 10$, $(34 + 0)^2 < (34 + i)^2 < (34 + 10)^2$

$1156 < (34 + i)^2 < 1936$

$1147 + 9 < (34 + i)^2 < 1147 + 789$

So $(34 + i)^2 \pmod{1147} = (34 + i)^2 - 1147$.

Our question is, for which $i$ does:

$((34 + i)^2 - 1147) \equiv 0 \pmod{2}$
The Quadratic Sieve: Example of dividing by 2

Need to know the set of $0 \leq i \leq 10$ such that 2 divides

$((34 + i)^2 \pmod{1147})$

What is $(34 + i)^2 \pmod{1147}$?
The Quadratic Sieve: Example of dividing by 2

Need to know the set of $0 \leq i \leq 10$ such that 2 divides

$$((34 + i)^2 \pmod{1147})$$

What is $(34 + i)^2 \pmod{1147}$? Since $0 \leq i \leq 10$,
The Quadratic Sieve: Example of dividing by 2

Need to know the set of $0 \leq i \leq 10$ such that 2 divides

$$((34 + i)^2 \pmod{1147})$$

What is $(34 + i)^2 \pmod{1147}$? Since $0 \leq i \leq 10$,

$$(34 + 0)^2 < (34 + i)^2 < (34 + 10)^2$$
The Quadratic Sieve: Example of dividing by 2

Need to know the set of $0 \leq i \leq 10$ such that 2 divides

$$((34 + i)^2 \pmod{1147})$$

What is $(34 + i)^2 \pmod{1147}$? Since $0 \leq i \leq 10,$

$$(34 + 0)^2 < (34 + i)^2 < (34 + 10)^2$$

$$1156 < (34 + i)^2 < 1936$$
The Quadratic Sieve: Example of dividing by 2

Need to know the set of $0 \leq i \leq 10$ such that 2 divides

$$((34 + i)^2 \pmod{1147})$$

What is $(34 + i)^2 \pmod{1147}$? Since $0 \leq i \leq 10$,

$$(34 + 0)^2 < (34 + i)^2 < (34 + 10)^2$$

$$1156 < (34 + i)^2 < 1936$$

$$1147 + 9 < (34 + i)^2 < 1147 + 789$$

So $(34 + i)^2 \pmod{1147} = (34 + i)^2 - 1147$. 
The Quadratic Sieve: Example of dividing by 2

Need to know the set of $0 \leq i \leq 10$ such that 2 divides

$$((34 + i)^2 \pmod{1147})$$

What is $(34 + i)^2 \pmod{1147}$? Since $0 \leq i \leq 10,$

$$(34 + 0)^2 < (34 + i)^2 < (34 + 10)^2$$

$$1156 < (34 + i)^2 < 1936$$

$$1147 + 9 < (34 + i)^2 < 1147 + 789$$

So $(34 + i)^2 \pmod{1147} = (34 + i)^2 - 1147.$

Our question is, for which $i$ does:

$$(34 + i)^2 - 1147 \equiv 0 \pmod{2}$$
The Quadratic Sieve: Example of dividing by 2, cont

Need to know the set of $0 \leq i \leq 10$ such that 2 divides

$((34 + i)^2 \pmod{1147})$
The Quadratic Sieve: Example of dividing by 2, cont

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We know that

$$(34 + i)^2 \pmod{1147} = (34 + i)^2 - 1147.$$
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Our question is, for which $i$ does:

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The Quadratic Sieve: Example of dividing by 2, cont

Need to know the set of $0 \leq i \leq 10$ such that 2 divides 

$((34 + i)^2 \pmod{1147})$

We know that 

$(34 + i)^2 \pmod{1147} = (34 + i)^2 - 1147$.

Our question is, for which $i$ does:

$(34 + i)^2 - 1147 \equiv 0 \pmod{2}$

Take mod 2 to both sides to get 

$i^2 - 1 \equiv 0 \pmod{2}$

Great!- just need to divide the $y_j$ where $i \equiv 1 \pmod{2}$. 
For which $0 \leq i \leq 10$ does $3$ divide $(34 + i)^2 \pmod{1147}$?
For which $0 \leq i \leq 10$ does $3$ divide $(34 + i)^2 \pmod{1147}$?
For which $0 \leq i \leq 10$ does $3$ divide $(34 + i)^2 \pmod{1147}$?

We know that $(34 + i)^2 \pmod{1147} = (34 + i)^2 - 1147$.

Our question is, for which $i$ does

$$(34 + i)^2 - 1147 \equiv 0 \pmod{3}$$

$$(1 + i)^2 - 1 \equiv 0 \pmod{3}$$

$$i \equiv 1, 2 \pmod{3}.$$ 

Great!- just need to divide the $y_i$ where $i \equiv 1, 2 \pmod{2}$. 

The Quadratic Sieve: Example of dividing by 3
The Quad Sieve: Example of dividing by 5, 7, 11, 13

\[(34 + i)^2 - 1147 \equiv 0 \pmod{5}\]
\[(4 + i)^2 - 2 \equiv 0 \pmod{5}\]
NO SOLUTIONS

\[(34 + i)^2 - 1147 \equiv 0 \pmod{7}\]
\[(6 + i)^2 \equiv 1 \pmod{7}\]
i \equiv 0, 2 \pmod{7}

\[(34 + i)^2 - 1147 \equiv 0 \pmod{11}\]
\[(1 + i)^2 \equiv 3 \pmod{11}\]
i \equiv 4, 5 \pmod{11}

\[(34 + i)^2 - 1147 \equiv 0 \pmod{13}\]
\[(8 + i)^2 + 10 \equiv 0 \pmod{13}\]
i \equiv 1, 9 \pmod{13}
The Quad Sieve: Example of dividing by 17,19,23

\[(34 + i)^2 - 1147 \equiv 0 \pmod{17}\]
\[i^2 + 9 \equiv 0 \pmod{17}\]
\[i \equiv 5, 12 \pmod{17}\]

\[(34 + i)^2 - 1147 \equiv 0 \pmod{19}\]
\[(15 + i)^2 + 12 \equiv 0 \pmod{19}\]
\[i \equiv 8, 15 \pmod{19}\]

\[(34 + i)^2 - 1147 \equiv 0 \pmod{23}\]
\[(11 + i)^2 + 3 \equiv 0 \pmod{23}\]
NO SOLUTIONS
The $B$-Factor Step Using Quad Sieve: Program

**Problem** Given $N$, $B$, $M$, $x$, want to $B$-factor

$$(x + 0)^2 \pmod{N}$$

$$
\vdots
$$

$$(x + M)^2 \pmod{N}$$

**Algorithm**

for all primes $p \leq B$

Find $A \subseteq \{0, \ldots, p - 1\}$: $i \in A$ iff $(x + i)^2 - N \equiv 0 \pmod{p}$

for $a \in A$

for $k = 0$ to $\left\lceil \frac{M-a}{p} \right\rceil$

divide $(x + pk + a)^2$ by $p$ (and then $p$ again...)

**Time**\medskip

\[ \leq \sum_{p \leq B} (\lg p + 2 \frac{M-1}{p}) = \sum_{p \leq B} \lg p + 2M \sum_{p \leq B} \frac{1}{p}. \]

\[ = (\sum_{p \leq B} \lg p) + 2M \ln \ln(B) = 2B + 2M \ln(\ln(B)). \]
Quad Sieve Alg: Second Attempt, Algorithm

Given \( N \) let \( x = \left\lceil \sqrt{N} \right\rceil \). All \( \equiv \) are mod \( N \). \( B, M \) are params.

\( B \)-factor \((x + 0)^2 \pmod{N}, \ldots, (x + M)^2 \pmod{M}\) by Quad S.

Let \( I \subseteq \{0, \ldots, M\} \) so that \((\forall i \in I), y_i \) is \( B \)-factored. Find \( J \subseteq I \) such that \( \sum_{i \in J} \vec{v}_i = \vec{0} \). Hence \( \prod_{i \in J} y_i \) has all even exponents.

\[
\prod_{i \in J} y_i = q_1^{2e_1} q_2^{2e_2} \cdots q_k^{2e_k}
\]

\[
(\prod_{i \in J}(x + i))^2 \equiv \left(\prod_{i \in J} q_i^{e_i}\right)^2 \pmod{N}
\]

Let \( X = \prod_{i \in J}(x + i) \) and \( Y = \prod_{i \in J} q_i^{e_i} \).

\[
X^2 - Y^2 \equiv 0 \pmod{N}.
\]

\( \text{GCD}(X - Y, N), \text{GCD}(X + Y, N) \) should yield factors.
Analysis of Quadratic Sieve Factoring Algorithm

Time to $B$-factor:

$$2B + 2M \ln(\ln(B)).$$

Time to find $J$: $B^3$.

Total Time:

$$2B + 2M \ln(\ln(B)) + B^3$$

Intuitive but not rigorous arguments yield run time

$$e^{\sqrt{\ln N \ln \ln N}} \sim e^{\sqrt{8 \ln N}} \sim e^{2.8\sqrt{\ln N}}$$
Speed Up One

Recall:

\[(34 + i)^2 - 1147 \equiv 0 \pmod{23}\]
\[(11 + i)^2 + 3 \equiv 0 \pmod{23}\]

NO SOLUTIONS
Speed Up One

Recall:
\[(34 + i)^2 - 1147 \equiv 0 \pmod{23}\]
\[(11 + i)^2 + 3 \equiv 0 \pmod{23}\]
NO SOLUTIONS

If there is a prime \(p\) such that \(z^2 \equiv 1147 \pmod{p}\) has NO SOLUTION then we should not ever consider it.
Speed Up One

Recall:
\[(34 + i)^2 \equiv -1147 \equiv 0 \pmod{23}\]
\[(11 + i)^2 + 3 \equiv 0 \pmod{23}\]

NO SOLUTIONS

If there is a prime \( p \) such that \( z^2 \equiv 1147 \pmod{p} \) has NO SOLUTION then we should not ever consider it.

There is a fast test to determine just if \( z^2 \equiv 1147 \pmod{p} \) has a solution (and more generally \( z^2 \equiv N \pmod{p} \)). So can eliminate some primes \( p \leq B \) before you start.
Recall:
We started with \( x = \left\lfloor \sqrt{N} \right\rfloor \) and did \((x + i)^2\) for \(0 \leq i \leq M\).
Recall:
We started with $x = \left\lceil \sqrt{N} \right\rceil$ and did $(x + i)^2$ for $0 \leq i \leq M$.

We can also (with some care) use $(x + i)^2$ when $i \leq 0$.

**Advantage** Smaller numbers more likely to be $B$-fact.
Speed Up Three

Recall:

\[(34 + i)^2 - 1147 \equiv 0 \pmod{19}\]
\[(15 + i)^2 + 12 \equiv 0 \pmod{19}\]
\[i \equiv 8, 15 \pmod{19}\]
Recall:
\[(34 + i)^2 - 1147 \equiv 0 \pmod{19}\]
\[(15 + i)^2 + 12 \equiv 0 \pmod{19}\]
\[i \equiv 8, 15 \pmod{19}\]

We can have one more variable:
\[(34j + i)^2 - 1147 \equiv 0 \pmod{19}\]
\[(15j + i)^2 + 12 \equiv 0 \pmod{19}\]
\[15j + i \equiv 8, 15 \pmod{19}\]
Many values of \((i, j)\) work.
1. Look at all of the non $B$-factored numbers. For each one test if what is left is prime. Let $X$ be the set of all of those primes.

2. Look at all of the non $B$-factored numbers. For each of them try a factoring algorithm (e.g., either of Pollards) for a limited amount of time. Let $Y$ be the set of primes you come across.

3. Do Q. Sieve on all of the non $B$-factored numbers using the primes in $X \cup Y$.

This will increase the number of $B$-factored numbers.
For this slide lg means ⌈lg⌉ which is very fast on a computer.

**Using Divisions** Primes \(q_1, \ldots, q_m < B\) divide \(x\). Divide \(x\) by all the \(q_i\). Also \(q_i^2, q_i^3, \text{ etc until does not work. When you are done you’ve } B\)-factored the number or not.
Speed Up Five—Avoid Division

For this slide lg means \( \lceil \log \rceil \) which is very fast on a computer.

**Using Divisions** Primes \( q_1, \ldots, q_m < B \) divide \( x \). Divide \( x \) by all the \( q_i \). Also \( q_i^2, q_i^3, \) etc until does not work. When you are done you’ve \( B \)-factored the number or not.

**Using Subtraction** Primes \( q_1, \ldots, q_m < B \) divide \( x \). Do

\[
d = \log(x) - \log(q_1) - \log(q_2) - \cdots - \log(q_m)
\]
Speed Up Five—Avoid Division

For this slide \(\lg\) means \(\lceil\lg\rceil\) which is very fast on a computer.

**Using Divisions** Primes \(q_1, \ldots, q_m < B\) divide \(x\). Divide \(x\) by all the \(q_i\). Also \(q_i^2, q_i^3, \text{etc} \) until does not work. When you are done you’ve \(B\)-factored the number or not.

**Using Subtraction** Primes \(q_1, \ldots, q_m < B\) divide \(x\). Do

\[
d = \lg(x) - \lg(q_1) - \lg(q_2) - \cdots - \lg(q_m)
\]

If \(d \sim 0\) then we think \(x\) IS \(B\)-fact, so \(B\)-factor \(x\).
If far from 0 then DO NOT DIVIDE!
Why Does This Work? If $x = q_1 q_2 q_3$ then

$$\lg(x) = \lg(q_1) + \lg(q_2) + \lg(q_3)$$

$$\lg(x) - \lg(q_1) - \lg(q_2) - \lg(q_3) = 0$$
Speed Up Five—Avoid Division, Why Works

Why Does This Work? If $x = q_1 q_2 q_3$ then

$$\lg(x) = \lg(q_1) + \lg(q_2) + \lg(q_3)$$

$$\lg(x) - \lg(q_1) - \lg(q_2) - \lg(q_3) = 0$$

So why not insist that

$$\lg(x) - \lg(q_1) - \lg(q_2) - \cdots - \lg(q_m) = 0$$

1. Using $\lceil \lg \rceil$ may introduce approximations so you don’t get 0.
2. If $x = q_1^2 q_2 q_3$ then

$$\lg(x) = \lg(q_1^2) + \lg(q_2) + \lg(q_3) = 2 \lg(q_1) + \lg(q_2) + \lg(q_3)$$

$$\lg(x) - \lg(q_1) + \lg(q_2) + \lg(q_3) = \lg(q_1) \neq 0$$

3. We need to define small carefully. Will still err.
Why is this fast?

1. Subtraction is much faster than division.
2. Most numbers are not $B$-fact, so don’t do divisions that won’t help.
Speed Up Five—Avoid Division, Example One

\[ B = 7 \text{ so we are looking at } 2, 3, 5, 7, 11, 13, 17. \text{ Small is } \leq 10. \]
人民法院—规避除法，例一

\( B = 7 \) 所以我们正在寻找 2, 3, 5, 7, 11, 13, 17。小数是 \( \leq 10 \)。

108290 7-fact? 我们发现 2, 5, 7, 13, 17 都能整除它。
$B = 7$ so we are looking at $2, 3, 5, 7, 11, 13, 17$. Small is $\leq 10$.

108290 7-fact? We find that $2, 5, 7, 13, 17$ all divide it.

$$\lg(108290) - \lg(2) - \lg(5) - \lg(7) - \lg(13) - \lg(17) = 4 \leq 10$$
$B = 7$ so we are looking at $2, 3, 5, 7, 11, 13, 17$. Small is $\leq 10$.

108290 7-fact? We find that $2, 5, 7, 13, 17$ all divide it.

$$\log(108290) - \log(2) - \log(5) - \log(7) - \log(13) - \log(17) = 4 \leq 10$$

So we think 108290 IS 7-fact. Is this correct? Yes:
\[ B = 7 \] so we are looking at 2, 3, 5, 7, 11, 13, 17. Small is \( \leq 10 \).

108290 7-fact? We find that 2, 5, 7, 13, 17 all divide it.

\[
\log(108290) - \log(2) - \log(5) - \log(7) - \log(13) - \log(17) = 4 \leq 10
\]

So we think 108290 IS 7-fact. Is this correct? Yes:

\[ 108290 = 2 \times 5 \times 7^2 \times 13 \times 17 \]
Speed Up Five—Avoid Division, Examples Two

Is 78975897 7-fact? We find that 3, 7, 11, 13, 17 all divide it.

\[
\log(78975897) - \log(3) - \log(7) - \log(11) - \log(13) - \log(17) = 11 > 10
\]

So we think 78975897 is NOT 7-fact. Is this correct? No!

\[
78975897 = 3 \times 7^2 \times 11 \times 13^2 \times 17^4
\]

Cautionary Note

78975897 = 3 \times 7^2 \times 11 \times 13^2 \times 17^4. was thought to NOT be 7-fact. Erred because primes had large exponents. The large exponents made \( \log(78975897) \) LARGER than \( \log(3) + \log(7) + \log(11) + \log(13) + \log(17) \).
Is 78975897 7-fact? We find that 3, 7, 11, 13, 17 all divide it.

\[ \lg(78975897) - \lg(3) - \lg(7) - \lg(11) - \lg(13) - \lg(17) = 11 > 10 \]

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78975897 = 3 × 7^2 × 11 × 13^2 × 17^4. was thought to NOT be 7-fact. Erred because primes had large exponents. The large exponents made \( \lg(78975897) \) LARGER than \( \lg(3) + \lg(7) + \lg(11) + \lg(13) + \lg(17) \).
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\[ \log(78975897) \]

LARGER than

\[ \log(3) + \log(7) + \log(11) + \log(13) + \log(17) \]
Is 9699690 7-fact? We find that 2, 3, 5, 7, 11, 13, 17 all divide it.
Is $9699690$ 7-fact? We find that $2, 3, 5, 7, 11, 13, 17$ all divide it.

$$\log(9699690) - \log(2) - \log(3) - \log(5) - \log(7) - \log(11) - \log(13) - \log(17) = 1 \leq 10$$
Is 9699690 7-fact? We find that 2,3,5,7,11,13,17 all divide it.

\[ \lg(9699690) - \lg(2) - \lg(3) - \lg(5) - \lg(7) - \lg(11) - \lg(13) - \lg(17) = 1 \leq 10 \]

So we think 9699690 is 7-fact. Is this correct? No!

\[ \lg(9699690) - \lg(2) - \lg(3) - \lg(5) - \lg(7) - \lg(11) - \lg(13) - \lg(17) = 1 \leq 10 \]

**Cautionary Note** 78975897 = 2 × 3 × 5 × 7 × 11 × 13 × 17 × 19. was thought to NOT be 7-fact. Erred because it had low exponents and only one a small prime over \(B\).
Is 9699690 7-fact? We find that 2, 3, 5, 7, 11, 13, 17 all divide it.

\[
\log(9699690) - \log(2) - \log(3) - \log(5) - \log(7) - \log(11) - \log(13) - \log(17) = 1 \leq 10
\]

So we think 9699690 is 7-fact. Is this correct? No!

\[
\log(9699690) - \log(2) - \log(3) - \log(5) - \log(7) - \log(11) - \log(13) - \log(17) = 1 \leq 10
\]

**Cautionary Note** 78975897 = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19. was thought to NOT be 7-fact. Erred because it had low exponents and only one a small prime over \( B \).

**Lemon to Lemonade** Not \( B \)-fact, but still useful. Speedup 4.
We are just approximating if

$$\lg x - \lg(q_1) - \cdots - \lg(q_m)$$

is small.
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\[ \lg x - \lg(q_1) - \cdots - \lg(q_m) \]

is small.

\( \lg 2, \lg 3, \lg 5 \) are so tiny, don’t bother with those.
We are just approximating if

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If $B = 7$ then use:

$$2^3, 3^2, 5^2, 7, 11, 13, 17, 19$$
Speed Up Six

The Gaussian Elimination is over $\mathbb{Z}_2$ and is for a sparse matrix (most of the entries are 0).

There are special purpose algorithms for this.

1. Can be done in $O(B^{2+\epsilon})$ steps rather than $O(B^3)$.
2. Can't store the entire matrix—to big.
(This is a paragraph from a blog post about Quad Sieve
https://blogs.msdn.microsoft.com/devdev/2006/06/19/
factoring-large-numbers-with-quadratic-sieve/)

Is $z$ $B$-fact? There is a light for each $p \leq B$ whose intensity is proportional to the $\log p$. Each light turns on just two times every $p$ cycles, corresponding to the two square roots of $N \mod p$. A sensor senses the combined intensity of all the lights together, and if this is close enough to the $\log z$ then $z$ is a $B$-fact number candidate. Can do in parallel.
The Quad Sieve had run time:

$$e^{(\ln N \ln \ln N)^{1/2}} \sim e^{2.8(\ln N)^{1/2}}$$
The Number Field Sieve

The Quad Sieve had run time:

$$e^{(\ln N \ln \ln N)^{1/2}} \sim e^{2.8(\ln N)^{1/2}}$$

The Number Field Sieve which uses some of the same ideas has run time:

$$e^{1.9(\ln N)^{1/3}(\ln \ln N)^{2/3}} \sim e^{14(\ln N)^{1/3}}$$
## Compare Run Times

<table>
<thead>
<tr>
<th>Alg</th>
<th>Run Time as $N^{a/L^{b}}$</th>
<th>Run Time in terms of $L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Naive</td>
<td>$N^{1/2}$</td>
<td>$2^{L/2}$</td>
</tr>
<tr>
<td>Pollard Rho</td>
<td>$N^{1/4}$</td>
<td>$2^{L/4}$</td>
</tr>
<tr>
<td>Linear Sieve</td>
<td>$N^{3.9/L^{1/2}}$</td>
<td>$2^{1.95L^{1/2}}$</td>
</tr>
<tr>
<td>Quad Sieve</td>
<td>$N^{2.8/L^{1/2}}$</td>
<td>$2^{1.4L^{1/2}}$</td>
</tr>
<tr>
<td>N.F. Sieve</td>
<td>$N^{14/L^{2/3}}$</td>
<td>$2^{20L^{1/3}}$</td>
</tr>
</tbody>
</table>

1. Times are more conjectured than proven.
2. Quad S. is better than Linear Sieve by only a constant in the exponent. Made a big difference IRL.
3. Quad Sieve is better than Pollard-Rho at about $10^{50}$. 
Relevance for RSA

2. People did not think it would work that well; however, he had friends at Sandia Labs who tried it out. Just for fun.
3. At the same time another group at Sandia Labs was working on a serious RSA project that would use 100-digit \( N \).
4. Quad Sieve could factor 100-digit numbers, so the RSA project had to be scrapped.
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The Future of Factoring

I paraphrase The Joy of Factoring by Wagstaff:
The best factoring algorithms have time complexity of the form

\[ e^{c(\ln N)^t(\ln \ln N)^{1-t}} \]

with Q.Sieve using \( c = \frac{1}{2} \) and N.F.Sieve using \( c = \frac{1}{3} \). Moreover, any method that uses \( B \)-factoring must take this long.
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  - We’ve run out of parameters to optimize.
  - Brandon, Solomon, Mark, and Ivan haven’t worked on it yet.