BILL START RECORDING

Pollard's ρ Algorithm for Factoring (1975)

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$\gcd(x-y, N)$ will likely yield a nontrivial factor of N since p divides both.

We look at several approaches to finding such an x, y that do not work before presenting the approach that does work.

Approach 1: Rand Seq mod p, Intuition

Generate random sequence $x_1, x_2, \ldots \in \{0, \ldots, N-1\}$.

Every time you get a new x_i, test, for all $1 \le j \le i-1$,

$$x_i \equiv x_j \pmod{p}.$$

Hope to get a YES.

If get YES then do

$$\gcd(x_i - x_j, N).$$

Approach 1: Rand Seq mod p, Program

$x_1 \leftarrow \mathrm{rand}(1, N-1)$, $i \leftarrow 2$
while TRUE
    $x_i \leftarrow \mathrm{rand}(1, N-1)$
        for $j \leftarrow 1$ to $i-1$
            if $x_i \equiv x_j \pmod{p}$ then
                $d \leftarrow \gcd(x_i - x_j, N)$
                if $d \neq 1$ and $d \neq N$ then break
    $i \leftarrow i+1$
output(d)

Approach 1: Rand Seq mod p, Program

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x_1 ← rand(1, N − 1), i ← 2
while TRUE
    x_i ← rand(1, N − 1)
      for j ← 1 to i − 1
        if x_i ≡ x_j (mod p) then
          d ← gcd(x_i − x_j, N)
          if d ≠ 1 and d ≠ N then break
    i ← i + 1
output(d)
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PRO: Bday paradox: x_i's are balls, mod p are boxes. So likely to find $x_i \equiv x_j \pmod{p}$ within $p^{1/2} \sim N^{1/4}$ iterations.

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ADJUST: Always do GCD.

Approach 2: Rand Seq mod p, W/O p, Intuition

Generate random sequence $x_1, x_2, \ldots \in \{0, \ldots, N-1\}$.

Every time you get a new x_i, do, for all $1 \le j \le i-1$,

$$\gcd(x_i - x_j, N).$$

So do not need to know p. And if $x_i \equiv x_j \pmod{p}$, you'll get a factor.

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x_1 ← rand(1, N − 1) i ← 2
while TRUE
    x_i ← rand(1, N − 1)
    for j ← 1 to i − 1
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CON: Iteration i makes i^2 operations. Total number of operations:

$$\sum_{i=1}^{N^{1/4}} i^2 \sim (N^{1/4})^3 \sim N^{3/4} \text{ BAD :-(.}$$

Another Issue: Space

$x_1 \leftarrow \mathrm{rand}(1, N-1)$ $i \leftarrow 2$
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CON: After Iteration i need to store $x_1, \ldots, x_i$. Since $\sim N^{1/4}$ iterations this is $N^{1/4}$ space. Too much space :-(

Approach 3: Rand Looking Sequence, Intuition

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- Pick random $x_1, c \in \{1, \ldots, N-1\}$.
- If know x_{i-1}, create

$$x_i = x_{i-1} * x_{i-1} + c \pmod{N}.$$

- The sequence x_1, x_2, x_3 will **hopefully** be random enough that the bday paradox applies. We use the informal term **random looking** for this.

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PRO Space not a problem.
CON Time still a problem :-(

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We want to find $i, j \leq N^{1/4}$ such that $x_i \equiv x_j \pmod{p}$.

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Lemma If exists $i < j \leq M$ with $x_i \equiv x_j$ then exists $k \leq M$ such that $x_k \equiv x_{2k}$.

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Important **If** there is a pair **then** there is a pair of form x_i, x_{2i}.

Idea Only try pairs of form (x_i, x_{2i}).

Final Algorithm

Define $f_c(x) \leftarrow x * x + c$

$x \leftarrow \mathrm{rand}(1, N-1)$, $c \leftarrow \mathrm{rand}(1, N-1)$, $y \leftarrow f_c(x)$
while TRUE
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PRO By Bday Paradox will likely finish in $N^{1/4}$ steps.

CON No real cons, but is $N^{1/4}$ fast enough?

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  - Proving that a deterministic sequence is **random enough** is hard to do or even define.
  - Erika, Guido, and Nataliya haven't worked on it yet.

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It works in practice, can we make it work in theory?

Why is it important to learn why it works in theory?

1. Make sure it really works. This is low-priority. Hey! It works!
2. If we know how it works in theory then perhaps can improve it. This is high-priority. Commonly theory and practice work together to improve both.

BILL STOP RECORDING