Public Key Cryptography
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2. Given \(n\), find a safe prime of length \(n\) and a generator \(g\).
3. Given \(a, b\) rel prime find inverse of \(a \mod b\): Euclidean alg.
Number Theory Assumptions

1. Discrete Log is hard.
2. Factoring is hard.

Note Actual hardness assumptions are not quite DL hard and Factoring hard but are close.
The Diffie-Helman Key Exchange

Alice and Bob will share a secret $s$. Security parameter $L$. 

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\[ \text{Hardness assumption: } f \text{ is hard to compute.} \]
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2. Alice sends $(p, g)$ to Bob in the clear (Eve sees $(p, g)$).

3. Alice picks random $a \in \{1, \ldots, p-1\}$, computes $g^a$ and sends it to Bob in the clear (Eve sees $g^a$).

4. Bob picks random $b \in \{1, \ldots, p-1\}$, computes $g^b$ and sends it to Alice in the clear (Eve sees $g^b$).

5. Alice computes $(g^b)^a = g^{ab}$.

6. Bob computes $(g^a)^b = g^{ab}$.

7. $g^{ab}$ is the shared secret.

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ElGamal Uses DH So Can Control Message

1. Alice and Bob do Diffie Helman.
2. Alice and Bob share secret $s = g^{ab}$.
3. Alice and Bob compute $(g^{ab})^{-1} \pmod{p}$.
4. To send $m$, Alice sends $c = mg^{ab}$
5. To decrypt, Bob computes $c(g^{ab})^{-1} \equiv mg^{ab}(g^{ab})^{-1} \equiv m$

We omit discussion of Hardness assumption (HW)
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   Alice finds $d$ such that $ed \equiv 1 \pmod{R}$.
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4. Alice broadcasts $(N, e)$. (Bob and Eve both see it.)
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5. **Bob**: To send $m \in \{1, \ldots, N - 1\}$, send $m^e \pmod{N}$. 
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4. Alice broadcasts $(N, e)$. (Bob and Eve both see it.)
5. Bob: To send $m \in \{1, \ldots, N - 1\}$, send $m^e \pmod{N}$.
6. If Alice gets $m^e \pmod{N}$ she computes

\[
(m^e)^d \equiv m^{ed} \equiv m^{ed} \pmod{R} \equiv m^1 \pmod{R} \equiv m
\]
Recall If Alice and Bob do RSA and Eve observes:

1. Eve sees $(N, e, m^e)$. The message is $m$.
2. Eve knows that there exist primes $p, q$ such that $N = pq$, but she does not know what $p, q$ are.
3. Eve knows that $e$ is relatively prime to $(p−1)(q−1)$.

**Definition:** Let $f$ be $f(N, e, m^e) = m$, where $N = pq$ and $e$ has an inverse mod $(p−1)(q−1)$.

**Hardness assumption (HA):** $f$ is hard to compute.
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Hardness Assumption for RSA

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Plain RSA Bytes!

The RSA given above is referred to as **Plain RSA**. Insecure! $m$ is always coded as $m^e \pmod{N}$.

**Note:** $rm$ means $r$ CONCAT with $m$ here and elsewhere.
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Make secure by padding: \( m \in \{0,1\}^{L_1} \), \( r \in \{0,1\}^{L_2} \).
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Make secure by padding: $m \in \{0, 1\}^{L_1}$, $r \in \{0, 1\}^{L_2}$.

To send $m \in \{0, 1\}^{L_1}$, pick rand $r \in \{0, 1\}^{L_2}$, send $(rm)^e$.

(NOTE- $rm$ means $r$ CONCAT with $m$ here and elsewhere.)

**DEC:** Alice finds $rm$ and takes rightmost $L_1$ bits.

**Caveat:** RSA still has issues when used in real world. They have been fixed. Maybe.
Attacks on RSA

1. Factoring Algs we did: Jevons, Pollard $\rho$, Pollard $p-1$,
2. Factoring Algs we didn’t do: Quad Sieve, Number Field Sieve.
3. Low-e attack, Same-$N$ attacks.
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These attacks tell Alice and Bob how to *up their game*.
Factoring Algorithms: Pollard $\rho$
Pollard ρ Algorithm

Define \( f_c(x) \leftarrow x \cdot x + c \). Looks random.

\[ x \leftarrow \text{RAND}(0, N-1), \ c \leftarrow \text{RAND}(0, N-1), \ y \leftarrow f_c(x) \]

while TRUE

\[ x \leftarrow f_c(x) \]
\[ y \leftarrow f_c(f_c(y)) \]
\[ d \leftarrow \text{GCD}(x - y, N) \]

if \( d \neq 1 \) and \( d \neq N \) then break

output\( (d) \)
Pollard $\rho$ Algorithm: Thought Exp

Let $\rho$ be the least prime that divides $N$. We do not know $\rho$. 
Let $p$ be the least prime that divides $N$. We do not know $p$.
The sequence $x, f_c(x), f(f_c(x)), \ldots$ is random-looking.
Pollard $\rho$ Algorithm: Thought Exp

Let $p$ be the least prime that div $N$. We do not know $p$.
The sequence $x, f_c(x), f(f_c(x)), \ldots$ is random-looking.
Put each element of the seq into its $\equiv$ class mod $p$. 

Caveat: Need the sequence to be truly random to prove it works. Don't have that, but it works in practice.
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By Bday Paradox $\exists$ 2 elements of the seq in same bucket within the first $2\sqrt{p} \leq 2N^{1/4}$ with high prob.
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Hence ($\exists x, y)[x \equiv y \pmod{p}]$ so $GCD(x - y, N) \neq 1$. 
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