BILL, RECORD LECTURE!!!!
Gen 2-letter Sub and Matrix Codes

October 9, 2021
Shift, Affine, Vig, Gen Sub, Easy to Crack

Shift, Affine, Vig all 1-letter substitutions. Freq cracked them.

An Idea Which History Passed By:

Def

Gen Sub 2-Cipher with perm $f$ on $\{0, \ldots, 25\}^2$.

1. Encrypt via $xy \rightarrow f(xy)$.
2. Decrypt via $xy \rightarrow f^{-1}(xy)$.

Why never used?

1. It was used but they kept it hidden!
2. The key length is roughly $26^2 \times 10 = 6760$ bits.
3. There was never a time when it was (a) easy to use, (b) hard to crack, and (c) better ciphers were not known.

Need bijection of $\{0, \ldots, 25\} \times \{0, \ldots, 25\}$ that is easy to use.
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The Matrix Cipher

**Def** Matrix Cipher. Pick $M$ a $2 \times 2$ matrix.
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**Encode:** Break text $T$ into blocks of 2, apply $M$ to each pair.

**Decode:** Do the same only with $M^{-1}$.

**OH!** is it easy to see if $M^{-1}$ exists? To find $M^{-1}$?

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Do you recognize the expression $ad - bc$?

Determinant!
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Inverse Matrix in $\mathbb{C}$ and inMods

\[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

1. Matrix $M$ over $\mathbb{C}$ has an inverse iff $ad - bc \neq 0$.

2. Matrix $M$ over Mod $n$ has an inverse iff $ad - bc$ is rel prime to $n$ iff $ad - bc$ has an inverse in Mod $n$.

3. Matrix $M$ over Mod 26 has an inverse iff $ad - bc$ is rel prime to 26 iff $ad - bc$ has no factors of 2 or 13 iff has an inverse in Mod 26.
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\[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

Good News:

1. Can test if \( M^{-1} \) exists, and if so find it, easily.
2. \( M \) small, so Key small.
3. Applying \( M \) or \( M^{-1} \) to a vector is easy computationally.

Bad News:

1. Eve can crack using frequencies of pairs of letters.
2. Eve can crack with brute force–Key space < \( 26^4 = 456976 \). Small.

So what to do?
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**Def** Pick $n \in \mathbb{N}$ and $M$ an $n \times n$ matrix with det rel prime to 26.

1. Encrypt via $\vec{x} \rightarrow M(\vec{x})$.
2. Decrypt via $\vec{y} \rightarrow M^{-1}(\vec{y})$

We’ll take $n = 8$. 

1. $M$ still small, so Key small.
2. Finding $M^{-1}$, mult by $M$ or $M^{-1}$ fast.
3. Eve cannot use brute force. Key Space is $\sim 26^{64} \sim 10^{90}$, Number of protons is $\sim 10^{79}$. (the number of non-invertible matrices is very small so $26^{64}$ is a good approximation).
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Lets Try Brute Force Even if Slow

1. Input $T$, a coded text.
2. For every $8 \times 8$ matrix $M$ over $\mathbb{Z}_{26}$,
   2.1 Test if $M$ is invertible. If not then goto next matrix.
   2.2 Decode $T$ into $T'$ using $M$.
   2.3 IF IS-ENGLISH($T'$) = YES then STOP and output $T'$, else goto next matrix $M$.

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Need to refine the question.
Assume $T$ is long and in normal English.
Assume Eve only has access to the ciphertext. VOTE:
1. Yes There is a clever way to do much better than $26^{64}$.
2. No and we can PROVE we can't do better with ciphertext-only.
3. Unknown to Science if we can do better with ciphertext-only.

Yes. We can crack in time $8 	imes 26^8$. 
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Can Crack in $8 \times 26^8$

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Better Idea: We take life one row at a time.
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**Better Idea:** We take life **one row at a time**.

**Example:** $3 \times 3$ matrix cipher. Decode Matrix $M$.

$$T = t_1 t_2 \cdots t_N \text{ each } t_i \text{ is 3-long}$$
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**Example:** $3 \times 3$ matrix cipher. Decode Matrix $M$.

$$T = t_1 t_2 \cdots t_N$$ each $t_i$ is 3-long

Guess the first row of $M$. Say:

$$\begin{pmatrix} 1 & 1 & 7 \\ * & * & * \\ * & * & * \end{pmatrix}$$
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The attack in the last slide went through every Matrix. Better Idea: We take life one row at a time. Example: 3 × 3 matrix cipher. Decode Matrix $M$.

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$$(m_1^1, m_2^1, m_3^1, \ldots, m_N^1)$$

is every third letter.
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$$(m^1_1, m^1_2, m^1_3, \ldots, m^1_N)$$

is every third letter. Can do IS-ENGLISH on it.
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Eve knows that Alice and Bob decode with $8 \times 8$ Matrix $M$. Ciphertext is

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IF IS-ENGLISH($T'$) = YES then $r_i = r$ and goto next $i$. Else goto the next $r$. 
Can Crack in $8 \times 26^8$

Eve knows that Alice and Bob decode with $8 \times 8$ Matrix $M$. Ciphertext is

$$T = t_1 t_2 \cdots t_N \quad t_i = t_i^1 \cdots t_i^8$$

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IF IS-ENGLISH($T'$)=YES then $r_i = r$ and goto next $i$. Else goto the next $r$.

$M$ is

$$
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
 r_1 & \cdots & r_n \\
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Takes $8 \times 26^8$ steps.
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More General $n$

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The row-by-row method takes $O(n26^n)$. 
Important Lesson

**Assume:** $26^{64}$ time is big enough to thwart Eve.
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4. After Alice and Bob learn the technique they have to up their parameters. This is a mild win for Eve in that A and B have to work harder.
The History of Cryptography in One Slide

1. Alice and Bob come up with a Crypto system (e.g., Matrix Cipher with $n = 8$).
2. Alice and Bob think it's uncrackable and have a "proof" that it is uncrackable (e.g., Eve HAS to go through all $26^{64}$ matrices).
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4. Lather, Rinse, Repeat.

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Assume Alice and Bob are using the Matrix Cipher with $n$ large (80 is large enough).

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Have to ask this carefully.

1. Plaintext is a very long text of normal English.
2. Eve sees the entire ciphertext $T$.
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Can Eve decode $T$?

- **Vote** ▶ Yes
  Eve can find the plaintext (or most of it).
- **Vote** ▶ No
  One can prove that Eve cannot find the plaintext.
- **Vote** ▶ Unknown to Science!
  (Hmmm—the NSA might know and is not telling.)

**Answer** Unknown to Science.

**Question** How come Matrix cipher is not used? Discuss.
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**Eve has much more information.**
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In reality

Eve has much more information.

Eve will have old messages and what they decoded to.
Example of What Eve Might Know

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Cracking Matrix Cipher

Example using $2 \times 2$ Matrix Cipher.
Eve learns that $(13,24)$ encrypts to $(3, 9)$. Hence:

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\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix}
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So

$13a + 24b = 3$

$13c + 24d = 9$

Two linear equations, Four variables

If Eve learns one more 2-letter message decoding then she will have
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