BILL, RECORD LECTURE!!!!

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Public Key Crypto: Math Needed and Diffie-Hellman
Private-Key Ciphers

What do the following all have in common?

1. Shift Cipher
2. Affine Cipher
3. Vig Cipher
4. General Sub
5. General 2-char sub
6. Matrix Cipher
7. One-time Pad
8. Other ciphers we studied
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Can Alice and Bob establish a key without meeting?
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Alice and Bob need to **meet!** (Hence **Private-Key**.)
Can Alice and Bob establish a key without meeting?
**Yes!** And that is the **key** to public-key cryptography.
A good crypto system is such that:

1. The computational task to encrypt and decrypt is easy.
2. The computational task to crack is hard.

Caveats

1. Hard to achieve info-theoretic hardness (One-time pad).
3. Can use hardness assumptions (e.g. factoring is hard).
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Difficulty of Problems Based on Length of Input

Hardness of a problem is measured by time-to-solve as a function of length of input.

Examples
1. Given a Boolean formula $\phi(x_1, \ldots, x_n)$, is there a satisfying assignment? Seems to require $2^{\Omega(n)}$ steps.
2. Polynomial vs Exp time is our notion of easy vs hard.
3. Factoring $n$ can be done in $O(\sqrt{n})$ time: Discuss. Easy! NO!!: $n$ is of length $\lg n + O(1)$ (henceforth just $\lg n$). $\sqrt{n} = 2^{0.5 \lg n}$. Exponential. Better (but still exp) algs known.

Upshot
For numeric problems length is $\lg n$.

Encryption requires:
- Alice and Bob can Enc and Dec in time $\leq O(\log n)$.
- Eve needs time $\geq c \cdot O(\log n)$ to crack.

What Counts
We count math operations as taking 1 step. This could be an issue with enormous numbers. We will work with mods so not a problem.
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**Upshot** For numeric problems length is $\lg n$. Encryption requires:

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Hardness of a problem is measured by time-to-solve as a function of **length of input**.

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Math Needed for Both Diffie-Hellman and RSA
Notation

Let $p$ be a prime.

1. $\mathbb{Z}_p$ is the numbers $\{0, \ldots, p - 1\}$ with mod add and mult.
2. $\mathbb{Z}_p^*$ is the numbers $\{1, \ldots, p - 1\}$ with mod mult.

Convention By prime we will always mean a large prime, so in particular, NOT 2. Hence we can assume $\frac{p-1}{2}$ is in $\mathbb{N}$. 
Exponentiation Mod $p$
Exponentiation Mod \( p \)

**Problem** Given \( a, n, p \) find \( a^n \pmod{p} \)
Exponentiation Mod $p$

**Problem** Given $a, n, p$ find $a^n \pmod{p}$

Even though we use $p$ and $p$ is always prime, our algorithm works for any natural $p$. 
Exponentiation Mod $p$: First Attempt

**Problem** Given $a, n, p$ find $a^n \pmod{p}$

1. $x_0 = a^0 = 1$
2. For $i = 1$ to $n$, $x_i = ax_{i-1} \pmod{p}$
3. Let $x = x_n$
4. Output $x$

Is this a good idea?
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**Discuss** How many steps were used to compute $a^n \pmod{p}$?

**Answer** $\sim n$. 
Exponentiation Mod $p$: Example of a Good Alg

Want $3^{64}$ (mod 101). All math is mod 101.

\[x^0 = 3\]
\[x^1 = x\]
\[x^2 \equiv 9.\] This is $3^2$ (mod 101).
\[x^2 = x^1 \equiv 9^2 \equiv 81.\] This is $3^4$ (mod 101).
\[x^3 = x^2 \equiv 81^2 \equiv 97.\] This is $3^8$ (mod 101).
\[x^4 = x^3 \equiv 97^2 \equiv 16.\] This is $3^{16}$ (mod 101).
\[x^5 = x^4 \equiv 16^2 \equiv 54.\] This is $3^{32}$ (mod 101).
\[x^6 = x^5 \equiv 54^2 \equiv 88.\] This is $3^{64}$ (mod 101).

So in 6 steps we got the answer!

Discuss: How many steps are used to compute $a^n$ (mod $p$)?

\[\sim \log_2 n.\]

But the above algorithm only seems to work if $n$ is a power of 2.

Discuss: What if $n$ is not a power of 2?
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Discuss How many steps are used to compute $a^n \pmod{p}$?
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\[ x_0 = 3 \]
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**Discuss** What if $n$ is *not* a power of 2?
A Review of Base 2

Say we want to do \( a^n \pmod{p} \).
A Review of Base 2

Say we want to do $a^n \pmod{p}$. Express $n$ in binary.
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**Upshot** If write $n$ as a sum of powers of 2 with 0,1 coefficients then $n$ is of the form
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**Upshot** If write $n$ as a sum of powers of 2 with 0,1 coefficients then $n$ is of the form

$$n = n_L 2^L + \cdots + n_1 2^1 + n_0 2^0 = \sum_{i=0}^{L} n_i 2^i$$

Where $L = \lfloor \lg(n) \rfloor$ and $n_i \in \{0, 1\}$. 
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Where $L = \lfloor \lg(n) \rfloor$ and $n_i \in \{0, 1\}$. Note that $L$ is one less than the number of bits needed for $n$. 

Repeated Squaring Algorithm

All math is mod $p$. 

1. Input $(a, n, p)$.

2. Convert $n$ to base 2: $n = \sum_{i=0}^{L} n_i 2^i$. ($L$ is $\lfloor \lg(n) \rfloor$)

3. $x_0 = a$.

4. For $i = 1$ to $L$, $x_i = x_{2i-1}$

5. (Now have $a_{n0} 2^{0}, \ldots , a_{nL} 2^{L}$) Answer is $a_{n0} \times \cdots \times a_{nL} 2^{L}$

Number of operations:

Number of MULTS in step 4: $\leq \lfloor \lg(n) \rfloor \leq \lg(n)$

Number of MULTS in step 5: $\leq L = \lfloor \lg(n) \rfloor \leq \lg(n)$

Total number of MULTS: $\leq 2 \lg(n)$.

More refined: $\lg(n) + \text{(number of 1's in binary rep of } n) - 1$
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Example on next page
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3. $x_0 = a$.
4. For $i = 1$ to $L$, $x_i = x_{i-1}^2$
5. (Now have $a^{n_0 2^0}, \ldots, a^{n_L 2^L}$) Answer is $a^{n_0 2^0} \times \cdots \times a^{n_L 2^L}$

Number of operations:
Number of MULTS in step 4: $\leq \lfloor \lg(n) \rfloor \leq \lg(n)$
Repeated Squaring Algorithm

All math is mod $p$.

1. Input $(a, n, p)$.
2. Convert $n$ to base 2: $n = \sum_{i=0}^{L} n_i 2^i$. ($L$ is $\lfloor \log(n) \rfloor$)
3. $x_0 = a$.
4. For $i = 1$ to $L$, $x_i = x_{i-1}^2$
5. (Now have $a^{n_0 2^0}, \ldots, a^{n_L 2^L}$) Answer is $a^{n_0 2^0} \times \ldots \times a^{n_L 2^L}$

Number of operations:
Number of **MULTS** in step 4: $\leq \lfloor \log(n) \rfloor \leq \log(n)$
Number of **MULTS** in step 5: $\leq L = \lfloor \log(n) \rfloor \leq \log(n)$
Repeated Squaring Algorithm

All math is mod $p$.

1. Input $(a, n, p)$.
2. Convert $n$ to base 2: $n = \sum_{i=0}^{L} n_i 2^i$. ($L$ is $\lfloor \lg(n) \rfloor$)
3. $x_0 = a$.
4. For $i = 1$ to $L$, $x_i = x_{i-1}^2$
5. (Now have $a^{n_0 2^0}, \ldots, a^{n_L 2^L}$) Answer is $a^{n_0 2^0} \times \cdots \times a^{n_L 2^L}$

Number of operations:
Number of **MULTS** in step 4: $\leq \lfloor \lg(n) \rfloor \leq \lg(n)$
Number of **MULTS** in step 5: $\leq L = \lfloor \lg(n) \rfloor \leq \lg(n)$
Total number of **MULTS** $\leq 2 \lg(n)$. 
Repeated Squaring Algorithm

All math is mod $p$.

1. Input $(a, n, p)$.
2. Convert $n$ to base 2: $n = \sum_{i=0}^{L} n_i 2^i$. ($L$ is $\lfloor \lg(n) \rfloor$)
3. $x_0 = a$.
4. For $i = 1$ to $L$, $x_i = x_{i-1}^2$
5. (Now have $a^{n_0 2^0}, \ldots, a^{n_L 2^L}$) Answer is $a^{n_0 2^0} \times \cdots \times a^{n_L 2^L}$

Number of operations:

Number of **MULTS** in step 4: $\leq \lfloor \lg(n) \rfloor \leq \lg(n)$
Number of **MULTS** in step 5: $\leq L = \lfloor \lg(n) \rfloor \leq \lg(n)$
Total number of **MULTS** $\leq 2 \lg(n)$.
More refined: $\lg(n) + (\text{number of 1's in binary rep of } n) - 1$
Repeated Squaring Algorithm

All math is mod $p$.

1. Input $(a, n, p)$.
2. Convert $n$ to base 2: $n = \sum_{i=0}^{L} n_i 2^i$. ($L$ is $\lfloor \lg(n) \rfloor$)
3. $x_0 = a$.
4. For $i = 1$ to $L$, $x_i = x_{i-1}^2$
5. (Now have $a^{n_02^0}, \ldots, a^{n_L2^L}$) Answer is $a^{n_02^0} \times \cdots \times a^{n_L2^L}$

Number of operations:

Number of **MULTS** in step 4: $\leq \lfloor \lg(n) \rfloor \leq \lg(n)$

Number of **MULTS** in step 5: $\leq L = \lfloor \lg(n) \rfloor \leq \lg(n)$

Total number of **MULTS** $\leq 2 \lg(n)$.

More refined: $\lg(n) + (\text{number of 1's in binary rep of } n) - 1$

Example on next page
Example of Exponentiation: $17^{265} \pmod{101}$

$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$
Example of Exponentiation: $17^{265}$ (mod 101)

$$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$$

$$17^2^0 \equiv 17 \ (0 \ steps)$$

This took $8 \sim \log(265)$ multiplications so far.

The next step takes only two multiplications:

$$17^{265} \equiv 17^{2^8} \times 17^{2^3} \times 17^{2^0} \equiv 84 \times 36 \times 17 \equiv 100$$

Point:

Step 2 took $< \log(265)$ steps since base-2 rep had few 1's.
Example of Exponentiation: $17^{265} \pmod{101}$

$$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$$

$17^{2^0} \equiv 17 \ (0 \text{ steps})$

$17^{2^1} \equiv 17^{2} \equiv 87 \ (1 \text{ step})$
Example of Exponentiation: $17^{265} \pmod{101}$

$$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$$

$17^{20} \equiv 17 \ (0 \text{ steps})$

$17^{21} \equiv 17^2 \equiv 87 \ (1 \text{ step})$

$17^{22} \equiv 87^2 \equiv 95 \ (1 \text{ step})$

This took $8 \approx \lg(265)$ multiplications so far.

The next step takes only two multiplications:

$$17^{265} \equiv 17^{28} \times 17^{23} \times 17^{20} \equiv 84 \times 36 \times 17 \equiv 100$$

Point:

Step 2 took $< \lg(265)$ steps since base-2 rep had few 1's.
Example of Exponentiation: $17^{265} \pmod{101}$

$$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$$

$17^{2^0} \equiv 17$ (0 steps)
$17^{2^1} \equiv 17^2 \equiv 87$ (1 step)
$17^{2^2} \equiv 87^2 \equiv 95$ (1 step)
$17^{2^3} \equiv 95^2 \equiv 36$ (1 step)

This took $8 \sim \lg(265)$ multiplications so far.

The next step takes only two multiplications:

$17^{2^{65}} \equiv 17^{2^8} \times 17^{2^3} \times 17^{2^0} \equiv 84 \times 36 \times 17 \equiv 100$ 

Point:
Step 2 took $< \lg(265)$ steps since base-2 rep had few 1's.
Example of Exponentiation: $17^{265} \pmod{101}$

$$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$$

$17^{2^0} \equiv 17 \ (0 \text{ steps})$

$17^{2^1} \equiv 17^2 \equiv 87 \ (1 \text{ step})$

$17^{2^2} \equiv 87^2 \equiv 95 \ (1 \text{ step})$

$17^{2^3} \equiv 95^2 \equiv 36 \ (1 \text{ step})$

$17^{2^4} \equiv 36^2 \equiv 84 \ (1 \text{ step})$

This took $\sim \lg(265)$ multiplications so far.

The next step takes only two multiplications:

$$17^{265} \equiv 17^{2^8} \times 17^{2^3} \times 17^{2^0} \equiv 84 \times 36 \times 17 \equiv 100$$

Point: Step 2 took $< \lg(265)$ steps since base-2 rep had few 1’s.
Example of Exponentiation: $17^{265} \pmod{101}$

$$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$$

$17^{2^0} \equiv 17 \ (0 \ steps)$

$17^{2^1} \equiv 17^2 \equiv 87 \ (1 \ step)$

$17^{2^2} \equiv 87^2 \equiv 95 \ (1 \ step)$

$17^{2^3} \equiv 95^2 \equiv 36 \ (1 \ step)$

$17^{2^4} \equiv 36^2 \equiv 84 \ (1 \ step)$

$17^{2^5} \equiv 84^2 \equiv 87 \ (1 \ step)$

This took $8 \sim \lg(265)$ multiplications so far.

The next step takes only two multiplications:

$17^{2^6} \equiv 87 \times 36 \times 17 \equiv 100$

Point: Step 2 took $< \lg(265)$ steps since base-2 rep had few 1's.
Example of Exponentiation: $17^{265} \pmod{101}$

$$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$$

$17^{2^0} \equiv 17 \; (0 \text{ steps})$
$17^{2^1} \equiv 17^2 \equiv 87 \; (1 \text{ step})$
$17^{2^2} \equiv 87^2 \equiv 95 \; (1 \text{ step})$
$17^{2^3} \equiv 95^2 \equiv 36 \; (1 \text{ step})$
$17^{2^4} \equiv 36^2 \equiv 84 \; (1 \text{ step})$
$17^{2^5} \equiv 84^2 \equiv 87 \; (1 \text{ step})$
$17^{2^6} \equiv 87^2 \equiv 95 \; (1 \text{ step})$

This took $8 \sim \lg(265)$ multiplications so far.

The next step takes only two multiplications:

$17^{265} \equiv 17^{2^8} \times 17^{2^3} \times 17^{2^0} \equiv 84 \times 36 \times 17 \equiv 100$

Point: Step 2 took $< \lg(265)$ steps since base-2 rep had few 1's.
Example of Exponentiation: \(17^{265} \pmod{101}\)

\[265 = 2^8 + 2^3 + 2^0 = (100001001)_2\]

\[
\begin{align*}
17^{2^0} &\equiv 17 \ (0 \text{ steps}) \\
17^{2^1} &\equiv 17^2 \equiv 87 \ (1 \text{ step}) \\
17^{2^2} &\equiv 87^2 \equiv 95 \ (1 \text{ step}) \\
17^{2^3} &\equiv 95^2 \equiv 36 \ (1 \text{ step}) \\
17^{2^4} &\equiv 36^2 \equiv 84 \ (1 \text{ step}) \\
17^{2^5} &\equiv 84^2 \equiv 87 \ (1 \text{ step}) \\
17^{2^6} &\equiv 87^2 \equiv 95 \ (1 \text{ step}) \\
17^{2^7} &\equiv 95^2 \equiv 36 \ (1 \text{ step})
\end{align*}
\]

This took \(\sim \log_2(265)\) multiplications so far.

The next step takes only two multiplications:

\[
17^{265} \equiv 17^{2^8} \cdot 17^{2^3} \cdot 17^{2^0} \equiv 84 \cdot 36 \cdot 17 \equiv 100
\]

Point:
Step 2 took \(< \log_2(265)\) steps since base-2 rep had few 1's.
Example of Exponentiation: \(17^{265} \pmod{101}\)

\[265 = 2^8 + 2^3 + 2^0 = (100001001)_2\]

\[
\begin{align*}
17^{2^0} & \equiv 17 \quad (0 \text{ steps}) \\
17^{2^1} & \equiv 17^2 \equiv 87 \quad (1 \text{ step}) \\
17^{2^2} & \equiv 87^2 \equiv 95 \quad (1 \text{ step}) \\
17^{2^3} & \equiv 95^2 \equiv 36 \quad (1 \text{ step}) \\
17^{2^4} & \equiv 36^2 \equiv 84 \quad (1 \text{ step}) \\
17^{2^5} & \equiv 84^2 \equiv 87 \quad (1 \text{ step}) \\
17^{2^6} & \equiv 87^2 \equiv 95 \quad (1 \text{ step}) \\
17^{2^7} & \equiv 95^2 \equiv 36 \quad (1 \text{ step}) \\
17^{2^8} & \equiv 36^2 \equiv 84 \quad (1 \text{ step})
\end{align*}
\]

This took \(8 \sim \lg(265)\) multiplications so far.

The next step takes only two multiplications:

\[
17^{2^8} \equiv 17^{2^6} \cdot 17^{2^2} \equiv 84 \cdot 36 \equiv 100
\]

Point:

Step 2 took \(\ll \lg(265)\) steps since base-2 rep had few 1's.
Example of Exponentiation: $17^{265} \pmod{101}$

$$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$$

$$17^{2^0} \equiv 17 \ (0 \text{ steps})$$
$$17^{2^1} \equiv 17^2 \equiv 87 \ (1 \text{ step})$$
$$17^{2^2} \equiv 87^2 \equiv 95 \ (1 \text{ step})$$
$$17^{2^3} \equiv 95^2 \equiv 36 \ (1 \text{ step})$$
$$17^{2^4} \equiv 36^2 \equiv 84 \ (1 \text{ step})$$
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This took $8 \sim \lg(265)$ multiplications so far.
Example of Exponentiation: $17^{265} \pmod{101}$

$$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$$

$17^{2^0} \equiv 17 \ (0 \text{ steps})$
$17^{2^1} \equiv 17^2 \equiv 87 \ (1 \text{ step})$
$17^{2^2} \equiv 87^2 \equiv 95 \ (1 \text{ step})$
$17^{2^3} \equiv 95^2 \equiv 36 \ (1 \text{ step})$
$17^{2^4} \equiv 36^2 \equiv 84 \ (1 \text{ step})$
$17^{2^5} \equiv 84^2 \equiv 87 \ (1 \text{ step})$
$17^{2^6} \equiv 87^2 \equiv 95 \ (1 \text{ step})$
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This took $8 \sim \lg(265)$ multiplications so far.

The next step takes only two multiplications:

$$17^{265} \equiv 17^{2^8} \times 17^{2^3} \times 17^{2^0} \equiv 84 \times 36 \times 17 \equiv 100$$
Example of Exponentiation: $17^{265} \pmod{101}$

$$265 = 2^8 + 2^3 + 2^0 = (100001001)_2$$

$17^{2^0} \equiv 17$ (0 steps)

$17^{2^1} \equiv 17^2 \equiv 87$ (1 step)

$17^{2^2} \equiv 87^2 \equiv 95$ (1 step)

$17^{2^3} \equiv 95^2 \equiv 36$ (1 step)

$17^{2^4} \equiv 36^2 \equiv 84$ (1 step)

$17^{2^5} \equiv 84^2 \equiv 87$ (1 step)

$17^{2^6} \equiv 87^2 \equiv 95$ (1 step)

$17^{2^7} \equiv 95^2 \equiv 36$ (1 step)

$17^{2^8} \equiv 36^2 \equiv 84$ (1 step)

This took $8 \sim \lg(265)$ multiplications so far.

The next step takes only two multiplications:

$$17^{2^{65}} \equiv 17^{2^8} \times 17^{2^3} \times 17^{2^0} \equiv 84 \times 36 \times 17 \equiv 100$$

**Point:** Step 2 took $\lg(265)$ steps since base-2 rep had few 1's.
Generators and Discrete Logarithms
Generators \((\text{mod } p)\)

Let’s take powers of 3 mod 7. All math is mod 7.

\[
3^1 \equiv 3 \\
3^2 \equiv 3 \times 3 \equiv 9 \equiv 2 \\
3^3 \equiv 3 \times 3^2 \equiv 3 \times 2 \equiv 6 \\
3^4 \equiv 3 \times 3^3 \equiv 3 \times 6 \equiv 18 \equiv 4 \\
3^5 \equiv 3 \times 3^4 \equiv 3 \times 4 \equiv 12 \equiv 5 \\
3^6 \equiv 3 \times 3^5 \equiv 3 \times 5 \equiv 15 \equiv 1
\]

\{3^1, 3^2, 3^3, 3^4, 3^5, 3^6\} = \{1, 2, 3, 4, 5, 6\}

Not in order.

3 is a generator for \(\mathbb{Z}^*_{7}\).
Generators \ (mod \ p)

Let’s take powers of 3 mod 7. All math is mod 7.
\[
3^1 \equiv 3
\]
Generators (mod $p$)

Let’s take powers of 3 mod 7. All math is mod 7.

$3^1 \equiv 3$

$3^2 \equiv 3 \times 3^1 \equiv 9 \equiv 2$
Generators (mod $p$)

Let’s take powers of 3 mod 7. All math is mod 7.

$3^1 \equiv 3$

$3^2 \equiv 3 \times 3^1 \equiv 9 \equiv 2$

$3^3 \equiv 3 \times 3^2 \equiv 3 \times 2 \equiv 6$
Let’s take powers of 3 mod 7. All math is mod 7.

$3^1 \equiv 3$

$3^2 \equiv 3 \times 3^1 \equiv 9 \equiv 2$

$3^3 \equiv 3 \times 3^2 \equiv 3 \times 2 \equiv 6$

$3^4 \equiv 3 \times 3^3 \equiv 3 \times 6 \equiv 18 \equiv 4$
Generators \ (mod \ p)

Let’s take powers of 3 mod 7. All math is mod 7.

\[3^1 \equiv 3\]
\[3^2 \equiv 3 \times 3^1 \equiv 9 \equiv 2\]
\[3^3 \equiv 3 \times 3^2 \equiv 3 \times 2 \equiv 6\]
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Generators \ (\text{mod} \ p) \ 

Let’s take powers of 3 mod 7. All math is mod 7.

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3^1 \equiv 3 \\
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3^5 \equiv 3 \times 3^4 \equiv 3 \times 4 \equiv 12 \equiv 5 \\
3^6 \equiv 3 \times 3^5 \equiv 3 \times 5 \equiv 15 \equiv 1
\]
Generators (mod $p$)

Let’s take powers of 3 mod 7. All math is mod 7.

$3^1 ≡ 3$
$3^2 ≡ 3 \times 3^1 ≡ 9 ≡ 2$
$3^3 ≡ 3 \times 3^2 ≡ 3 \times 2 ≡ 6$
$3^4 ≡ 3 \times 3^3 ≡ 3 \times 6 ≡ 18 ≡ 4$
$3^5 ≡ 3 \times 3^4 ≡ 3 \times 4 ≡ 12 ≡ 5$
$3^6 ≡ 3 \times 3^5 ≡ 3 \times 5 ≡ 15 ≡ 1$

$\{3^1, 3^2, 3^3, 3^4, 3^5, 3^6\} = \{1, 2, 3, 4, 5, 6\}$ Not in order.
Generators \ (mod \ p)

Let's take powers of 3 mod 7. All math is mod 7.

\[\begin{align*}
3^1 & \equiv 3 \\
3^2 & \equiv 3 \times 3^1 \equiv 9 \equiv 2 \\
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\end{align*}\]

\[\{3^1, 3^2, 3^3, 3^4, 3^5, 3^6\} = \{1, 2, 3, 4, 5, 6\}\] Not in order.

3 is a **generator** for \(\mathbb{Z}_7^*\).
Generators (mod $p$)

Let’s take powers of 3 mod 7. All math is mod 7.

$3^1 \equiv 3$

$3^2 \equiv 3 \times 3^1 \equiv 9 \equiv 2$

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$$\{3^1, 3^2, 3^3, 3^4, 3^5, 3^6\} = \{1, 2, 3, 4, 5, 6\} \text{ Not in order.}$$

3 is a generator for $\mathbb{Z}_7^*$. 

**Definition:** If $p$ is a prime and $\{g^1, \ldots, g^{p-1}\} = \{1, \ldots, p - 1\}$

then $g$ is a generator for $\mathbb{Z}_p^*$. 
Fact: 3 is a generator mod 101. All math is mod 101. Discuss the following with your neighbor:

1. Find \( x \) such that \( 3^x \equiv 81 \).

2. Find \( x \) such that \( 3^x \equiv 92 \).
   Try computing \( 3^1, 3^2, \ldots \) until you get 92. Might take \( \sim 100 \) steps. Shortcut?

3. Find \( x \) such that \( 3^x \equiv 93 \).
   Try computing \( 3^1, 3^2, \ldots \) until you get 93. Might take \( \sim 100 \) steps. Shortcut?

2nd and 3rd look hard. Are they? VOTE Both hard, both easy, one of each, unknown to science.

3 \( x \) \( \equiv \) 92 easy. 3 \( x \) \( \equiv \) 93 Not known how hard.
Discrete Log-Example

Fact: 3 is a generator mod 101. All math is mod 101.

Discuss the following with your neighbor:

1. Find $x$ such that $3^x \equiv 81$. $x = 4$ obv works.

2. Find $x$ such that $3^x \equiv 92$. Try computing $3^1, 3^2, \ldots$, until you get 92. Might take $\sim 100$ steps. Shortcut?

3. Find $x$ such that $3^x \equiv 93$. Try computing $3^1, 3^2, \ldots$, until you get 93. Might take $\sim 100$ steps. Shortcut?

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Discuss the following with your neighbor:
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2. Find $x$ such that $3^x \equiv 92$. 

Might take $\sim 100$ steps.

Shortcut?

3. Find $x$ such that $3^x \equiv 93$. 

Try computing $3^1, 3^2, \ldots$, until you get 93.

Might take $\sim 100$ steps.

Shortcut?

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   - Try computing $3^1, 3^2, \ldots$, until you get 92.
   - Might take $\sim 100$ steps.

3. Find $x$ such that $3^x \equiv 93$.
   - Try computing $3^1, 3^2, \ldots$, until you get 93.
   - Might take $\sim 100$ steps.
Discrete Log-Example

**Fact:** 3 is a generator mod 101. All math is mod 101.

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3. Find $x$ such that $3^x \equiv 93$.
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   - Might take $\sim 100$ steps. Shortcut?

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   Might take $\sim 100$ steps. Shortcut?

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   Might take $\sim 100$ steps.
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Discuss the following with your neighbor:

1. Find $x$ such that $3^x \equiv 81$. $x = 4$ obv works.
2. Find $x$ such that $3^x \equiv 92$.
   Try computing $3^1, 3^2, \ldots$, until you get 92.
   Might take $\sim 100$ steps. Shortcut?
3. Find $x$ such that $3^x \equiv 93$.
   Try computing $3^1, 3^2, \ldots$, until you get 93.
   Might take $\sim 100$ steps. Shortcut?

2nd and 3th look hard. Are they?
VOTE Both hard, both easy, one of each, unknown to science.

$3^x \equiv 92$ easy. $3^x \equiv 93$ Not known how hard.
Discrete Log-Example: \(3^x \equiv 92 \pmod{101}\)

**Fact:** 3 is a generator mod 101. All math is mod 101.
Discrete Log-Example: $3^x \equiv 92 \pmod{101}$

**Fact:** 3 is a generator mod 101. All math is mod 101. Find $x$ such that $3^x \equiv 92$. Easy!
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Fact: 3 is a generator mod 101. All math is mod 101. Find $x$ such that $3^x \equiv 92$. Easy!

1. $92 \equiv 101 - 9 \equiv (-1)(9) \equiv (-1)3^2$.
2. $3^{50} \equiv -1$ (WHAT! Really?)
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**Fact:** 3 is a generator mod 101. All math is mod 101. Find $x$ such that $3^x \equiv 92$. Easy!

1. $92 \equiv 101 - 9 \equiv (-1)(9) \equiv (-1)3^2$.
2. $3^{50} \equiv -1$ (WHAT! Really?)
3. $92 \equiv 3^{50} \times 3^2 \equiv 3^{52}$. So $x = 52$ works.
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Generalize:
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Generalize:

1. If $g$ is a generator of $\mathbb{Z}_p^*$ then $g^{(p-1)/2} \equiv p - 1 \equiv -1$. 
Discrete Log-Example: $3^x \equiv 92 \pmod{101}$

**Fact:** 3 is a generator mod 101. All math is mod 101. Find $x$ such that $3^x \equiv 92$. Easy!

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3. $92 \equiv 3^{50} \times 3^2 \equiv 3^{52}$. So $x = 52$ works.

Generalize:

1. If $g$ is a generator of $\mathbb{Z}_p^*$ then $g^{(p-1)/2} \equiv p - 1 \equiv -1$.
2. So finding $x$ such that $g^x \equiv p - g^a \equiv -g^a$ is as easy as $g^a$.

$$x = \frac{p - 1}{2} + a : \quad g^{\frac{p-1}{2} + a} = g^{\frac{p-1}{2}} g^a \equiv -g^a$$
Discrete Log-Example: $3^x \equiv 93 \pmod{101}$

**Fact:** 3 is a generator mod 101. All math is mod 101. Is there a trick for $g^x \equiv 93 \pmod{101}$? Not that I know of.
Formally Discrete Log is.

**Def** The **Discrete Log (DL)** problem is as follows:

1. Input $g, a, p$. With $1 \leq g, a \leq p - 1$. $g$ is a gen for $\mathbb{Z}_p^\times$.
2. Output $x$ such that $g^x \equiv a \pmod{p}$.

Recall ▶ A good alg would be time $O(\log p)$.
▶ A bad alg would be time $p^{O(1)}$.
▶ If an algorithm is in time (say) $p^{1/10}$ still not efficient but will force Alice and Bob to up their game.
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Bill’s Opinion on DL. Also Applies to Factoring

1. Fact: DL in QuantumP.

2. BILL Opinion: Quantum computers that can do DL fast won’t happen in my lifetime. In your lifetime. Ever.

3. Fact: Good classical algorithms using hard number theory exist and have been implemented. Still exponential but low constants. Some are amenable to parallelism.

4. BILL Opinion: The biggest threat to crypto is from hard math combined with special purpose parallel hardware.

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Kunal’s Opinion on DL. Also Applies to Factoring

1. Quantum computing has the power to break modern crypto. But
2. Quantum computing is like graphene: quantum can do everything except leave the lab. See https://en.wikipedia.org/wiki/Graphene for information on graphene which seems to always be 5 years away from applications.

So Kunal and Bill agree.
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Josh’s Opinion on DL. Also Applies to Factoring

1. **PRO**
   A lot of corporations and governments are putting a lot of money into Quantum, so (unlike other alt-computing ideas) this one really has a shot.

2. **CON**
   The error-correction problem still seems hard.

3. **CONCLUSION**
   The question **When will quantum computers be able to really do DL fast** should be asked to physicists, not to CMSC/ENEE/MATH TAs.

Bill
Since lots of money is being put into it, if it does not work they won’t have the excuse that other technologies have of not having been tried.
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**Bill** Since lots of money is being put into it, if it does not work they won’t have the excuse that other technologies have of not having been tried.
Sajjad Opinion on DL. Also Applies to Factoring

Sajjad is working in Quantum Computing so
Sajjad Opinion on DL. Also Applies to Factoring

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1. He actually knows stuff.
Sajjad Opinion on DL. Also Applies to Factoring

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2. His funding depends on the answer being YES Quantum is practical!
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His opinion is on the next slide.
1. There has been huge progress in the last 10 years in error-correction.
2. There are now real problems (sampling) that can be done by a Quantum Computer much faster than a Classical Computer. Yeah!
3. The sampling problem is not related to problems in number theory like Discrete Log. Boo!
4. Prediction in ∼ 25 years we will have real quantum computers that can do DL and factoring quickly. This is based on what experimentalists say (see next slide). Bill Its very hard to predict things especially about the future.
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Expert Opinions on the Technical Realization of Quantum Computers
Expert Opinion As a Paper

See also this paper: qtime.pdf
Discrete Log-General

Definition Let \( p \) be a prime and \( g \) be a generator mod \( p \).
The Discrete Log Problem:
Given \( a \in \{1, \ldots, p\} \), find \( x \) such that \( g^x \equiv a \pmod{p} \). We call this \( DL_{p,g}(a) \).
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Given $a \in \{1, \ldots, p\}$, find $x$ such that $g^x \equiv a \pmod{p}$. We call this $DL_{p,g}(a)$.

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4. **Tradeoff:** By restricting $a$ we are cutting down search space for Eve. Even so, in this case we need to since she REALLY can recognize when DL is easy.
Consider What We Already Have Here

Exponentiation mod p is Easy.

Discrete Log is thought to be Hard.

We want a crypto system where:

Alice and Bob do Exponentiation mod p to encrypt and decrypt.

Eve has to do Discrete Log to crack it.

Do we have this?

No. But we’ll come close.
Consider What We Already Have Here

▶ Exponentiation mod \( p \) is Easy.
Consider What We Already Have Here

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- **Discrete Log** is thought to be Hard.

We want a crypto system where:

- Alice and Bob do **Exponentiation mod** $p$ to encrypt and decrypt.
- Eve has to do **Discrete Log** to crack it.

Do we have this?

No. But we’ll come close.
Convention

For the rest of the slides on **Diffie-Hellman Key Exchange** there will always be a prime $p$ that we are considering.

**ALL** math done from that point on is mod $p$.

**ALL** numbers are in $\{1, \ldots, p - 1\}$. 
Finding Generators
Finding Gens; How Many Gens Are There?

**Problem** Given $p$, find $g$ such that

- $g$ generates $\mathbb{Z}_p^*$.  
- $g \in \{\frac{p}{3}, \ldots, \frac{2p}{3}\}$. (We ignore floors and ceilings for notational convenience.)
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**How many elts of $\{1, \ldots, p - 1\}$ are gens?** $\Theta(\frac{p}{\log \log p})$
Finding Gens; How Many Gens Are There?

**Problem** Given \( p \), find \( g \) such that

- \( g \) generates \( \mathbb{Z}_p^* \).
- \( g \in \{ \frac{p}{3}, \ldots, \frac{2p}{3} \} \). (We ignore floors and ceilings for notational convenience.)

We could test \( \frac{p}{3} \), then \( \frac{p}{3} + 1 \), etc. Will we hit a generator soon?

**How many els of \( \{1, \ldots, p-1\} \) are gens?** \( \Theta \left( \frac{p}{\log \log p} \right) \)

Hence if you just look for a gen you will find one soon.
Finding Gens: First Attempt

Given prime \( p \), find a gen for \( \mathbb{Z}_p^* \)
Finding Gens: First Attempt

Given prime $p$, find a gen for $\mathbb{Z}_p^*$

1. Input $p$. 

PRO
You will find a gen fairly soon.

CON
Computing $g_1, \ldots, g_{p-1}$ is $O(p \log p)$ operations.

Bad!
Recall $(\log p)$ is fast, $O(p)$ is slow.
Given prime $p$, find a gen for $\mathbb{Z}_p^*$

1. Input $p$.
2. For $g = \frac{p}{3}$ to $\frac{2p}{3}$:
   
   Compute $g^1, g^2, \ldots, g^{p-1}$ until either hit a repeat or finish. If repeats then $g$ is NOT a generator, so goto the next $g$. If finishes then output $g$ and stop.

**PRO** You will find a gen fairly soon.
Finding Gens: First Attempt

**Given prime** \( p \), **find a gen for** \( \mathbb{Z}_p^* \)

1. Input \( p \).

2. For \( g = \frac{p}{3} \) to \( \frac{2p}{3} \):
   
   *Compute* \( g^1, g^2, \ldots, g^{p-1} \) *until either hit a repeat or finish.*  
   *If repeats then* \( g \) *is NOT a generator, so goto the next* \( g \).  
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**PRO** You will find a gen fairly soon.

**CON** Computing \( g^1, \ldots, g^{p-1} \) is \( O(p \log p) \) operations.
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**Given prime** $p$, find a gen for $\mathbb{Z}_p^*$

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**PRO** You will find a gen fairly soon.

**CON** Computing $g^1, \ldots, g^{p-1}$ is $O(p \log p)$ operations.

**Bad!** Recall $(\log p)^{O(1)}$ is fast, $O(p)$ is slow.
Theorem: If \( g \) is not a generator then there exists \( x \) that (1) \( x \) divides \( p - 1 \), (2) \( x \neq p - 1 \), and (3) \( g^x \equiv 1 \).

**Given prime** \( p \), find a gen for \( \mathbb{Z}_p^* \)
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**Given prime $p$, find a gen for $\mathbb{Z}_p^*$**

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---

**BIG CON:** Factoring $p - 1$? Really? Borrow Sajjad's Quantum Computer?
Finding Gens: Second Attempt

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1. Input \( p \).
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**Given prime $p$, find a gen for $\mathbb{Z}_p^*$**

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   *Compute $g^x$ for all $x \in F$. If any $= 1$ then $g$ not generator.*
   
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Is this a good algorithm?
Theorem: If $g$ is not a generator then there exists $x$ that 
(1) $x$ divides $p - 1$, (2) $x \neq p - 1$, and (3) $g^x \equiv 1$.

Given prime $p$, find a gen for $\mathbb{Z}^*_p$

1. Input $p$.
2. Factor $p - 1$. Let $F$ be the set of its factors except $p - 1$.
3. For $g = \frac{p}{3}$ to $\frac{2p}{3}$:
   Compute $g^x$ for all $x \in F$. If any $= 1$ then $g$ not generator. 
   If none are 1 then output $g$ and stop.

Is this a good algorithm?

Time Every iteration takes $O(|F| \cdot (\log p))$ ops. $|F|$ might be huge!
So no good. But wait for next slide...
Finding Gens: Second Attempt

Theorem: If \( g \) is not a generator then there exists \( x \) that
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BIG CON: Factoring \( p - 1 \)?
Finding Gens: Second Attempt

**Theorem:** If $g$ is *not* a generator then there exists $x$ that 
(1) $x$ divides $p - 1$, 
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**Given prime $p$, find a gen for $\mathbb{Z}_p^*$**

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**BIG CON:** Factoring \( p - 1 \)? Really?
Borrow Sajjad’s Quantum Computer?
Factoring is Hard. Or is it?

Second Attempt had two problems:

1. Factoring is hard.
2. $p - 1$ may have many factors.

We want $p - 1$ to be easy to factor and have few factors.

There are three kinds of people in the world:

1. Those who make things happen.
2. Those who watch things happen.
3. Those who wonder what happened.

We will make things happen.
We will make $p - 1$ easy to factor.
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Finding Gens: Third Attempt

**Idea:** Pick \( p \) such that \( p - 1 = 2q \) where \( q \) is prime.
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Given prime $p$, find a gen for $\mathbb{Z}_p^*$
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Given prime $p$, find a gen for $\mathbb{Z}_p^*$

1. Input $p$ a prime such that $p - 1 = 2q$ where $q$ is prime. (We later explore how we can find such a prime.)
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Is this a good algorithm?

**PRO:** Every iteration does $O(\log p)$ operations.

**CON:** Need both $p$ and $p - 1$ are primes.

**CAVEAT:** We need to pick certain kinds of primes. Can do that!
Finding Gens: Third Attempt

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