The Complexity of Grid Coloring

Daniel Apon—U of MD William Gasarch—U of MD
Kevin Lawler—Permanent

Grid Coloring

Notation: If $n \in \mathrm{N}$ then $[n]$ is the set $\{1, \ldots, n\}$.

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $G_{n,m}$ is c-colorable if there is a c-coloring of $G_{n,m}$ such that no rectangle has all four corners the same color.
2. $\chi(G_{n,m})$ is the least c such that $G_{n,m}$ is c-colorable.

Examples

A FAILED 2-Coloring of $G_{4,4}$

$$
\begin{array}{cccc}
R & B & B & R \\
B & R & R & B \\
B & B & R & R \\
R & R & R & B
\end{array}
$$

A 2-Coloring of $G_{4,4}$

$$
\begin{array}{cccc}
R & B & B & R \\
B & R & R & B \\
B & B & R & R \\
R & B & R & B
\end{array}
$$

Example: a 3-Coloring of G(10,10)

EXAMPLE: A 3-Coloring of $G_{10,10}$

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

It is known that CANNOT 2-color $G_{10,10}$. Hence $\chi(G_{10,10}) = 3$.

Obstruction Sets

Fenner-Gasarch-Glover-Purewall [FGGP] showed:

1. For all c there exists OBS_c, a finite set of grids, such that

   $$G_{n,m} \text{ is } c\text{-colorable iff no element of } OBS_c \text{ is inside } G_{m,n}.$$

2. FGGP have a proof which shows $|OBS_c| \leq 2c^2$.
3. If OBS_c is known then the set of c-colorable grids is completely characterized.

OBS-2 and OBS-3 Known

FGGP showed

$$\begin{aligned} OBS_2 &= \{G_{3,7}, G_{5,5}, G_{7,3}\} \\ OBS_3 &= \{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\} \end{aligned}$$

2-colorability table. C for Colorable, U for Uncolorability.

	2	3	4	5	6	7
2	C	C	C	C	C	C
3	C	C	C	C	C	U
4	C	C	C	C	C	U
5	C	C	C	U	U	U
6	C	C	C	U	U	U
7	C	U	U	U	U	U

4-Colorability

1. FGGP did not (as of 2009) determine OBS_4.
2. FGGP had reasons to think $G_{17,17}$ is 4-colorable but they did not have a 4-coloring.
3. In 2009 Gasarch offered a prize of \$289.00 for the first person to email him a 4-coloring of $G_{17,17}$.
4. Brian Hayes, Scientific American Math Editor, popularized the challenge.

Challenge Was Hard

1. Lots of people worked on it.
2. No progress.
3. Finally solved in 2012 by Bernd Steinbach and Christian Posthoff [SP]. Clever, and SAT-solver, but did not generalize.
4. They and others also found colorings that lead to $\mathrm{OBS}_4 = \{$

$$G_{5,41}, G_{6,31}, G_{7,29}, G_{9,25}, G_{18,23}, G_{11,22}, G_{13,21}, G_{17,19},$$
$$G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,18}, G_{22,11}, G_{21,13}, G_{19,17}$$

$\}$

Is Grid Coloring Hard?

We view this two ways:

1. Is there an NP-complete problem lurking here somewhere? YES!
2. Is there a Prop Statement about Grid Coloring whose resolution proof requires exp size? YES!

THERE IS AN NP-COMPLETE PROBLEM LURKING!

Grid Coloring Hard!-NP stuff

1. Let $c, N, M \in \mathbb{N}$. A partial mapping χ of $N \times M$ to $\{1, \ldots, c\}$ is a *extendable to a c-coloring* if there is an extension of χ to a total mapping which is a c-coloring of $N \times M$.
2. $$GCE = \{(N, M, c, \chi) \mid \chi \text{ is extendable}\}.$$

   GCE is NP-complete!

GCE is NP-complete

$\phi(x_1, \ldots, x_n) = C_1 \wedge \cdots \wedge C_m$ is a 3-CNF formula. We determine N, M, c and a partial c-coloring χ of $N \times M$ such that

$$\phi \in \text{3-SAT iff } (N, M, c, \chi) \in GCE$$

Forcing a Color to Only Appear Once in Main Grid

G								
G								
R		G						
G								
G								
G								
G	G	G	G	G	G	G	G	G

G can only appear once in the main grid, where it is, but what about R? (The double lines are not part of the construction. They are there to separate the main grid from the rest.)

Forcing a Color to Only Appear Once in Main Grid

R	G								
R	G								
R	R		G						
R	G								
R	G								
R	G								
R	G	G	G	G	G	G	G	G	G
R	G	R	R	R	R	R	R	R	R

G can only appear once in the main grid, where it is. R cannot appear anywhere in the main grid.

Using Variables

D means that the color is some *distinct*, unique color.

	D	D	D	D	D	D	D	D	D	D	D
$\overline{x}_1$		D	D	D	D	D	D	D	D	T	F
x_1		D	D	D	D	D	D	T	F	T	F
$\overline{x}_1$		D	D	D	D	T	F	T	F	D	D
x_1		D	D	T	F	T	F	D	D	D	D
$\overline{x}_1$		T	F	T	F	D	D	D	D	D	D
x_1		T	F	D	D	D	D	D	D	D	D

The labeled $x_1, \overline{x}_1$ are not part of the grid. They are visual aids.

Coding a Clause

$C_1 = L_1 \vee L_2 \vee L_3$. Where L_1, L_2, L_3 are literals (vars or their negations).

	$\cdots$	D	T	T
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_1	$\cdots$		D	F
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_2	$\cdots$			
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_3	$\cdots$		F	D
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The L_1, L_2, L_3 are not part of the grid. They are visual aids.

Coding a Clause—More Readable

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1		D	F
L_2			
L_3		F	D

One can show that

- If put any of TTT, TTF, TFT, FTT, FFT, FTF, TFF in first column then can extend to full coloring.
- If put FFF in first column then cannot extend to a full coloring.

Example: (F,F,T)

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1	F	D	F
L_2	F		$*$
L_3	T	F	D

The * is forced to be T.

Example: (F,F,T)

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1	F	D	F
L_2	F	$*$	T
L_3	T	F	D

The * is forced to be F.

Example: (F,F,T)

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1	F	D	F
L_2	F	F	T
L_3	T	F	D

Other Assignments

1. We did (F, F, T).
2. (F, T, F), (T, F, F) are similar.
3. (F, T, T), (T, F, T), (T, T, F), (T, T, T) are easier.

Cannot Use (F,F,F)

$C_1 = L_1 \vee L_2 \vee L_3$. Want that (F, F, F) CANNOT be extended to a coloring.

	D	T	T
L_1	F	D	F
L_2	F	$*$	$*$
L_3	F	F	D

The *'s are forced to be T.

Cannot Use (F,F,F)

	D	T	T
L_1	F	D	F
L_2	F	T	T
L_3	F	F	D

There is a mono rectangle of T's. NOT a valid coloring!

Put it all Together

Do the above for all variables and all clauses to obtain the result that GRID EXT is NP-complete!

Big Example

$$(x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_1 \vee \overline{x}_3 \vee \overline{x}_4)$$

												C_1	C_1	C_2	C_2	C_3	C_3
	D	D	D	D	D	D	D	D	D	D	D	T	T	T	T	T	T
$\overline{x}_4$		D	D	D	D	D	D	D	D	T	F	D	D	D	D	D	F
x_4		D	D	D	D	D	D	D	D	T	F	D	D	D	F	D	D
$\overline{x}_3$		D	D	D	D	D	D	T	F	D	D	D	D	D	D	D	D
x_3		D	D	D	D	T	F	T	F	D	D	D	D			D	D
$\overline{x}_3$		D	D	D	D	T	F	D	D	D	D	D	F	D	D		
$\overline{x}_2$		D	D	T	F	D	D	D	D	D	D	D	D	F	D	D	D
x_2		D	D	T	F	D	D	D	D	D	D			D	D	D	D
$\overline{x}_1$		T	F	D	D	D	D	D	D	D	D	D	D	D	D	F	D
x_1		T	F	D	D	D	D	D	D	D	D	F	D	D	D	D	D

Does this Explain why the Challenge was Hard?

1. MAYBE NOT: GCE is Fixed Parameter Tractable: For fixed c GCE_c is in time $O(N^2M^2 + 2^{O(c^4)})$. But for $c = 4$ this is huge!
2. MAYBE NOT: Our result says nothing about the case where the grid is originally all blank.

Open Questions

1. Improve Fixed Parameter Tractable algorithm.
2. NPC results for mono squares? Other shapes?
3. Show that

$$\{(n, m, c) : G_{n,m} \text{ is } c\text{-colorable } \}$$

is hard.

- If n, m in unary then sparse set, not NPC—New framework for hardness needed.
- If n, m binary then not in NP. Could try to prove *NEXP*-complete. But we the difficulty of the problem is not with the grid being large, but with the number-of-possibilities being large.