Lower Bounds on Resolution Theorem Proving Via Games (An Exposition)

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1. Stays Jukna’s book on Circuit complexity had the material.
3. Result itself is old; however this proof is new and wonderful.
Problem: Given a CNF-Formula $\varphi \notin SAT$ we want a proof that $\varphi \notin SAT$.

1. Need to define logical system rigorously.
2. Research Program: Show that in various Logic Systems cannot get a short proof.
RESOLUTION RULE

\[ A \lor x \quad B \lor \neg x \]

\[ \hline \]

\[ A \lor B \]
Definition
Let $\varphi = C_1 \land \cdots \land C_L$ be a CNF formula. A *Resolution Proof that* $\varphi \notin SAT$, *is a sequence of clauses such that on each line you have either*

1. One of the $C$’s in $\varphi$ (called an AXIOM).
2. $A \lor B$ where on prior lines you had $A \lor x$ and $B \lor \neg x$. Variable that is *resolved on* is $x$.
3. The last line has the empty clause.

**EASY:** If there is a Resolution Proof that $\varphi \notin SAT$ then $\varphi \notin SAT$. 
Example

\[ \varphi = x_1 \land x_2 \land (\neg x_1 \lor \neg x_2) \]

1. \( x_1 \) (AXIOM)
2. \( \neg x_1 \lor \neg x_2 \) (AXIOM)
3. \( \neg x_2 \) (From lines 1,2, resolve on \( x_1 \).)
4. \( x_2 \) (AXIOM)
5. \( \emptyset \) (From lines 3,4, resolve on \( x_2 \).)

DO IN CLASS ON BOARD AND THEN DO MORE EXAMPLES
Another Example

The AND of the following:

1. \( x_{11} \lor x_{12} \)
2. \( x_{21} \lor x_{22} \)
3. \( x_{31} \lor x_{32} \)
4. \( \neg x_{11} \lor \neg x_{21} \)
5. \( \neg x_{11} \lor \neg x_{31} \)
6. \( \neg x_{21} \lor \neg x_{31} \)
7. \( \neg x_{12} \lor \neg x_{22} \)
8. \( \neg x_{12} \lor \neg x_{32} \)
9. \( \neg x_{22} \lor \neg x_{32} \)
Another Example

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6. \( \neg x_{21} \lor \neg x_{31} \)
7. \( \neg x_{12} \lor \neg x_{22} \)
8. \( \neg x_{12} \lor \neg x_{32} \)
9. \( \neg x_{22} \lor \neg x_{32} \)

This is Pigeonhole Principle: \( x_{ij} \) is putting \( i \)th pigeon in \( j \) hole!
Another Example

The AND of the following:

1. $x_{11} \lor x_{12}$
2. $x_{21} \lor x_{22}$
3. $x_{31} \lor x_{32}$
4. $\neg x_{11} \lor \neg x_{21}$
5. $\neg x_{11} \lor \neg x_{31}$
6. $\neg x_{21} \lor \neg x_{31}$
7. $\neg x_{12} \lor \neg x_{22}$
8. $\neg x_{12} \lor \neg x_{32}$
9. $\neg x_{22} \lor \neg x_{32}$

This is Pigeonhole Principle: $x_{ij}$ is putting $i$th pigeon in $j$ hole! Can’t put 3 pigeons into 2 holes! DO RES PROOF IN CLASS.
Let $n < m$. $n$ is NUMBER OF HOLES, $m$ is NUMBER OF PIGEONS. $x_{ij}$ will be thought of as Pigeon $i$ IS in Hole $j$.

**Definition**

$PHP_n^m$ is the AND of the following:

1. For $1 \leq i \leq m$
   
   $$x_{i1} \lor x_{i2} \lor \cdots \lor x_{in}$$

   (Pigeon $i$ is in SOME Hole.)

2. For $1 \leq i_1 < i_2 \leq n$ and $1 \leq j \leq m$

   $$\neg x_{i1j} \lor \neg x_{i2j}$$

   (Hole $j$ does not have BOTH Pigeon $i_1$ and Pigeon $i_2$.)

**NOTE:** $PHP_n^m$ has $nm$ VARS and $mn^2$ CLAUSES.
An Assignment is an $m \times n$ array of 0’s and 1’s.

**Example:** $m = 4$, $n = 3$.

\[
\begin{array}{ccc}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{array}
\]

$x_{12} = x_{23} = x_{13} = x_{42} = 1$. All else 0. Violates PHP since have $x_{12} = x_{42} = 1$. 
1) Have two 1’s in a column.

```
   0 1 0
   0 0 1
   1 0 0
   0 1 0
```

2) Have an all 0’s row.

```
   0 1 0
   0 0 1
   0 0 0
   1 0 0
```
\[ \varphi(x_1, \ldots, x_v) = C_1 \land \cdots \land C_L \]

If \( \varphi \not\in \text{SAT} \) then construct Resolution Proof as follows:

1. Form a **DECISION TREE** with nodes on level \( i \) labeled \( x_i \).
2. Every leaf is a complete assignment. Output least indexed clause \( C \) that is 0.
3. Turn Decision Tree **UPSIDE DOWN**, its a Res. Proof. **DO EXAMPLE IN CLASS**
4. **NOTE:** Can always do roughly \( 2^v \) size proof.
5. **NOTE:** The Resolution Proofs are **TREE-Resolution**.
1. Informally- a Tree Resolution proof is one where if written out looks like a tree.

2. Formally- a Tree Resolution proof is one where any clause in the proof is used at most once.
Assume $n < m$.

1. $PHP^m_n$ always has a size roughly $2^{nm}$ Tree Resolution Proof.
2. We show $2^{n/2}$ size is REQUIRED. THIS IS POINT OF THE TALK!!!!!! (Better is known- roughly $2^{n \log n}$, but that is slightly harder.)
3. The lower bound is IND of $m$.
4. There is an upper bound of roughly $2^{n \log n}$: Resolution and the weak pigeonhole principle, By Buss and Pitassi. Proceedings of the 1997 Computer Science Logic Conference.
Parameters of the game: $p \in \mathbb{N}$,

$$\varphi = C_1 \land \cdots \land C_L \notin SAT.$$ 

Do the following until a clause is proven false:

1. **PROVER** picks a variable $x$ that was not already picked.
2. **DEL** either
   2.1 Sets $x$ to 0 or 1, OR
   2.2 Defers to **PROVER** .
      2.2.1 If **PROVER** sets $x = 0$ then **DEL** gets one points.
      2.2.2 If **PROVER** sets $x = 1$ then **DEL** gets one points.

At end if **DEL** has $p$ points then he **WINS**; otherwise **PROVER** **WINS**. HAVE THEM PLAY THE GAME WITH PHP.
We assume that PROVER and DEL play perfectly.

1. **PROVER wins** means **PROVER has a winning strategy**.
2. **DEL wins** means **DEL has a winning strategy**.
Lemma

Let $p \in \mathbb{N}$, $\varphi \notin \text{SAT}$. If $\varphi$ has a Tree Res proof of size $< 2^p$ then PROVER wins.

Proof.

PROVER Strategy:

1. Initially $T$ is res tree of size $< 2^p$ and DEL has 0 points.
2. PROVER picks $x$, the LAST var resolved on.
3. If DEL sets $x$ DEL gets no points.
4. If DEL defers then PROVER sets to 1 or 0- whichever yields a smaller tree. NOTE: One of the trees will be of size $< 2^{p-1}$. DEL gets 1 point.
5. Repeat: after $i$th stage will always have $T$ of size $< 2^{p-i}$, and DEL has $\leq i$ points.
Recall:

Lemma
Let $p \in \mathbb{N}$, $\varphi \notin SAT$. If $\varphi$ has a Tree Res proof of size $< 2^p$ then PROVER wins.

Contrapositive:

Lemma
Let $p \in \mathbb{N}$, $\varphi \notin SAT$. If DEL wins then EVERY Tree Resolution proof for $\varphi$ has size $\geq 2^p$.

PLAN: Get AWESOME strategy for DEL when $\varphi = PHP^m_n$. 

Lemma

Let $n \geq 2$. Let $n < m$. Let $\varphi = \text{PHP}^m_n$. There is a strategy for \text{DEL} that earns at least $\frac{n}{4}$ points.

KEY to STRATEGY FOR \text{DEL}:

1. \text{DEL} does NOT allow two 1’s in a column. EVER!!!!

2. \text{DEL} is wary of the all-0’s row. But not too wary. \text{DEL} puts a 1 in a row if PROVER has put many 0’s in that row.
PROVER has picked $x_{ij}$.

1. If there is a $i'$ such that $x_{i',j} = 1$ then set $x_{i,j} = 0$. (DEL gets no points, but averts DISASTER.)

2. If the $i$th row has $\frac{n}{2}$ 0's that PROVER put there, and no 1's, then DEL puts a 1 (DEL gets no points, but DEL delays an all-0 row.)

3. Otherwise defer to PROVER (and get some points!).
ANALYSE STRATEGY

Games over when some row is ALL 0’s—say row $i$.

$$x_{i1} = x_{i2} = \cdots = x_{in} = 0.$$ 

WHO set them to 0? There are two cases, though the second yields more cases.

1. **PROVER** set $\geq \frac{n}{2}$ of the vars to 0. Then **DEL** gets $\geq \frac{n}{2}$ points. DONE!

2. **DEL** set $\geq \frac{n}{2}$ of the vars to 0. See next two slides.
**ANALYSE STRATEGY-DELAYER SET** \( \geq \frac{n}{2} \) **VARS TO 0**

\( DEL \) set \( \geq \frac{n}{2} \) of the vars to 0. There is only ONE reason \( DEL \) every sets a var to 0—when it was set there was a 1 in that column. So \( \frac{n}{2} \) of the columns have a 1 in them. WHO set them to 1?

1. \( PROVER \) set \( \geq \frac{n}{4} \) of those vars to 1. Then \( DEL \) gets \( \geq \frac{n}{4} \) points. DONE.

2. \( DEL \) set \( \geq \frac{n}{4} \) of those vars to 1. See next slide.
DEL set \( \geq \frac{n}{4} \) of the vars to 1. There is only ONE reason DEL every sets a var to 1—there are \( \frac{n}{2} \) vars in that row set to 0 by PROVER. So each of the \( \frac{n}{4} \) vars that DEL set to 1 imply \( \frac{n}{2} \) 0’s set by PROVER which implies \( \frac{n}{2} \) points for DEL. So DEL gets \( \geq \frac{n^2}{8} \) points. DONE.
DEL has winning strategy to get

$$\min\left\{ \frac{n}{2}, \frac{n}{4}, \frac{n^2}{8} \right\}$$

points. Since $n \geq 2$ this min is $\frac{n}{4}$. 